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Certain Transformations and Summations of Basic Hypergeometric Series

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Abstract. In the present work we have established some new transformations and summations of basic hypergeometric series by making the use of WP-Bailey pairs.Using multiple q-integrals and a determinant evaluation, we establish a multivariable extension of Bailey's nonterminating is99 transformation. From this result, we deduce new multivariable terminating 1049 transformations, s&summations and other identities. We also use similar methods to derive new multivariable r+t summations. Some of our results are extended to the case of elliptic hypergeometric series.

Keywords: Bailey's lemma; Basic hypergeometric series; Transformation; Summation.

Introduction

For $|q| < 1, (a;q)_n = (1-a)(1-aq)\ldots\ldots\ldots(1-aq^n-1);n = 1,2\ldots.$

$$(a;q)_0 = 1; \qquad (a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n)$$

$$(a;q)_n = \frac{(a;q)_\infty}{(aqn\,;q)_\infty}\ .$$

where a is real or complex.

A Basic Hypergeometric Series is defined as

$$r\phi_s(a_1,a_2,a_3\ldots.a_r\,;b_1,b_2,b_3\ldots\ldots b_s\,;q,z) = \sum_{n=0}^{\infty}\frac{(a_1;q)_n(a_2;q)_n\ldots(a_r\,;q)_n}{(q;q)n(b_1;q)n\,(b_2;q)n.\ldots(b_s\,;q)_n}\ [(-1)^n q^{\frac{n(n-1)}{2}}\ ]^{1+s-r}z^n\ .$$

For 0 < |q| < 1, the series converges absolutely for all z if r ≤ s and for |z| < 1 if r = s+1. This series also converges absolutely if |q| > 1 and |z| < |b1b2...b_s |/|a_1a_2...a_r |.

In 1944, Bailey [1] introduced a very useful and simple identity known as Bailey's lemma. The Bailey's lemma states that, if

$$\beta n = \sum_{r=0}^{n}\alpha_r u_{n-r}v_{n+r}$$

$$\gamma_n = \sum_{r=n}^{\infty}\delta_r u_{r-n}v_{n+r}\ ,$$

then under the suitable convergence conditions and if change in the order of summations is allowed

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$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n \ ,$$

where α_r ,δ_r ,u_r and υ_r are functions of r such that β_n and γ_n exist. The proof of the lemma is trivial.

Taking $u_r = \frac{1}{(q;q)_r}$ *and* $v_r = \frac{1}{(aq;q)_r}$ *in* (1.1), *we have*

$$\beta n = \sum_{r=0}^{n} \frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}} \quad .$$

The pair of sequence (α_n,β_n) that satisfies (1.4) is called a Bailey pair relative to the parameter a.

The Bailey lemma has been a simple and effective tool in proving Rogers-Ramanujan type of identities and also a verity of transformations of basic hypergeometric series [2]. Slater [3, 4] used Bailey's lemma and gave the long list of 130 identities of Roger-Ramanujan type. After Slater the Bailey lemma have been extensively used to prove Rogers-Ramanujan type of identities and its generalizations [5-8]. Very recently, Warnaar [9] has written a very elegant survey of Bailey lemma. Andrews et al [10-13] exploited very effective the mechanism of Bailey's transform in the form of Bailey pair and Bailey chain. In particular, WP-Bailey pair (α_n,β_n) [14] satisfying

$$\beta_n = \sum_{r=0}^{n} \frac{(k/a;q)_{n-r}(k;q)_{n+r}}{(q;q)n-r(aq;q)n+r} \ \alpha r \ .$$

For k = 0 in (1.5), we get the standard Bailey pair (1.4). The relation (1.5) follows by setting $u_r = \frac{(k/a;q)_r}{(q;q)_r}$ *and* $v_r = \frac{(k;q)_r}{(aq;q)_r}$ in (1.1). The same substitutions in (1.2), gives

$$\gamma_n = \frac{(k;q)_{2n}}{(aq;q)_{2n}} \sum_{r=0}^{\infty} \frac{(k/a;q)_r (kq2n\,;q)_r}{(q;q)_r (aq2n+1\,;q)_r} \ \delta_{r+n}.$$

In the present paper, we have established a number of transformations and summations of basic hypergeometric series by making use of (1.5) and (1.6). Some interesting special cases have also been deduced.

We define a WP-Bailey Unit Bailey pair as

$$\alpha_n = \frac{(a, q\sqrt{a}, -q\sqrt{a}, a/k;q)_n}{(q, \sqrt{a}, -\sqrt{a}, kq;q)_n} (k/a)_n \ ,$$

$$\beta_n = \begin{matrix} 1, n = 0, \\ 0, n > 0. \end{matrix}$$

The trivial WP-Bailey pair is defined as

$$\beta_n = \frac{(k, k/a;q)_n}{(q, aq;q)_n} \ ,$$

$$\alpha_n = \begin{matrix} 1, n = 0, \\ 0, n > 0. \end{matrix}$$

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A WP-Bailey pair due to Singh [15] is

$$\alpha_n = \frac{(a, q\sqrt{a}, -q\sqrt{a}, y, z, a^2 q/kyz; q)n}{(q, \sqrt{a}, -\sqrt{a}, aq/y, aq/z, kyz/a; q)n} (k/a)^n ,$$

$$\beta_n = \frac{(ky/a, kz/a, k, aq/yz; q)_n}{(q, aq/y, aq/z, kyz/a; q)_n} .$$

In our analysis we shall also require the following known results,

$$4\phi_3(a, -q\sqrt{a}, b, c; -\sqrt{a}, aq/b, aq/c; q, q\sqrt{a}/bc) = \frac{(aq, q\sqrt{a}/b, q\sqrt{a}/c, aq/bc; q)_\infty}{(aq/b, aq/c, q\sqrt{a}, q\sqrt{a}/bc; q)_\infty}$$

$$3\phi_2(a, \lambda q, b; \lambda, q\lambda^2 /b; q, \lambda^2 /ab^2) = \frac{1-\lambda + \lambda/b(1-\lambda/a)(\lambda^2/b^2, q\lambda^2/ab; q)_\infty}{(1-\lambda)(1+\lambda/b)(q\lambda^2/b, \lambda^2/ab^2; q)_\infty} ,$$

$|\lambda^2 /ab^2| < 1.$

$$2\phi 1(a, b; aq/b; q, -q/b) = \frac{(-q;q)\infty(aq, aq^2/b^2; q^2)_\infty}{(-q/b, aq/b; q)_\infty} .$$

$$4\phi 3(a, q\sqrt{a}, -q\sqrt{a}, b; \sqrt{a}, -\sqrt{a}, aq/b; q, 1/b^2 q) = \frac{(a/b^2, 1/bq; q)_\infty}{(aq/b, 1/b^2 q; q)_\infty} .$$

$$8\phi 7(a, q\sqrt{a}, -q\sqrt{a}, \sqrt{(a/b)}, -\sqrt{(a/b)}, \sqrt{(aq/b)}, -\sqrt{(aq/b)}, b; \sqrt{a}, -\sqrt{a}, q\sqrt{(ab)}, -q\sqrt{(ab)}, \sqrt{(abq)}, -\sqrt{(abq)}, aq/b; q; bq) = \frac{(aq, b^2 q; q)\infty}{(bq, abq; q)\infty} ,$$

$$|bq| < 1.$$

Result and Discussion

If (α_n, β_n) is a WP-Bailey pair, then under suitable convergence conditions, the following relations are true

$$\sum_{n=0}^{\infty} \frac{(-q\sqrt{k}, c; q)_n}{(-\sqrt{k}, kq/c; q)_n} \left(\frac{aq}{c\sqrt{k}}\right)^n \beta_n =$$

$$\frac{(kq, aq/\sqrt{k}, q\sqrt{k}/c, aq/c; q)\infty}{(aq/c\sqrt{k}, aq, kq/c, q\sqrt{k}; q)\infty} \sum_{n=0}^{\infty} \frac{(k;q)_{2n}}{(kq;q)_{2n}} \frac{(-q\sqrt{k}, q\sqrt{k}, c; q)_n}{(-\sqrt{k}, aq/\sqrt{k}, aq/c; q)_n} \left(\frac{aq}{c\sqrt{k}}\right)^n \alpha_n.$$

$$\sum_{n=0}^{\infty} \frac{(q\, n+1\sqrt{ak}; q)n}{(q\, n\sqrt{ak}; q)n} \left(\frac{a^2}{k^2}\right)^n \beta_n =$$

$$\frac{(a/k, a^2 q/k; q)_\infty}{(a\, 2/k^2, aq; q)_\infty} \sum_{n=0}^{\infty} \frac{(k, kqn, q\, n+1\sqrt{ak}; q)_n}{(q\, n\sqrt{ak}, a\, 2q/k, a\, 2q\, n+1/k; q)_n} \frac{(1-\sqrt{akq}^{2n} + \sqrt{a}/(\sqrt{k} - a^{3/2}q^{2n}))}{(1-q^{2n}\sqrt{(ak)})(1+\sqrt{a}/\sqrt{k})} \left(\frac{a^2}{k^2}\right)^n \alpha_n.$$

$$\sum_{n=0}^{\infty}\frac{(q\sqrt{k},-q\sqrt{k};q)_n}{(\sqrt{k},-\sqrt{k};q)_n}\left(\frac{a^2}{k^2q}\right)^n\beta_n =$$

$$\frac{(a^2/k,a/kq;q)\infty}{(aq,a^2/k^2q;q)\infty}\sum_{n=0}^{\infty}\frac{(k,kq^n,q\sqrt{k},-q\sqrt{k};q)_n}{(a^2/k,a^2q\,n/k,\sqrt{k},-\sqrt{k};q)_n}\left(\frac{a^2}{qk^2}\right)^n\alpha_n.$$

$$\sum_{n=0}^{\infty}\frac{(q\sqrt{k},-q\sqrt{k},\sqrt{a},-\sqrt{a},\sqrt{aq},-\sqrt{aq};q)_n}{(\sqrt{k},-\sqrt{k},-kq\sqrt{(1/a)},kq\sqrt{(1/a)},k\sqrt{(q/a)},-k\sqrt{(q/a)};q)_n}\left(\frac{kq}{a}\right)^n\beta_n =$$

$$\frac{(kq,\ k^2q\ /\ a^2\ ;q)_\infty}{(kq\ /\ a,k^2q\ /\ a;q)_\infty}\sum_{n=0}^{\infty}\frac{\left(k,kq^n\ ,q\sqrt{k},-q\sqrt{k},\sqrt{a},-\sqrt{a},\sqrt{(aq)},-\sqrt{(aq)};q\right)_n}{\left(\sqrt{k},-\sqrt{k},kq\sqrt{(1\ /\ a)},-kq\sqrt{(1\ /\ a)},k\sqrt{(q\ /\ a)},-k\sqrt{(q\ /\ a)};q\right)_n}$$

$$\frac{(k^2\,q/a,k^2q^{(n+1)}/a;q)_n}{(aq,aq^{n+1},kq,kq^{n+1};q)_n}\left(\frac{kq}{a}\right)^n\alpha_n$$

$$\sum_{n=0}^{\infty}\left(\frac{-aq}{k}\right)^n\beta_n = \frac{(kq,a^2q^2/k;q^2)_\infty(-q;q)_\infty}{(-aq/k,aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(k,kq^n\,;q)_n}{(kq,a^2q^2/k;q^2)_{2n}}\left(\frac{-aq}{k}\right)^n\alpha_n.$$

Proof 2.1. Substituting $a = kq^{2n}, b = k/a\ and\ c = cq^n$ in (1.10), we have

$$4\phi3(kq^{2n},-q^{n+1}\sqrt{k},k/a,cq^n;-q^n\sqrt{k},aq^{2n+1},kq^{n+1}/c;q,aq/c\sqrt{k})$$

$$= \frac{(kq,aq/\sqrt{k},q\sqrt{k}/c,aq/c;q)_\infty(aq;q)_{2n}(kq/c,q\sqrt{k};q)_n}{(aq/c\sqrt{k},aq,kq/c,q\sqrt{k};q)_\infty(kq;q)_{2n}(aq/\sqrt{k},aq/c;q)_n}.$$

Putting $\delta_r = \frac{(c,-q\sqrt{k};q)r}{(-\sqrt{k},kq/c;q)r}\left(\frac{aq}{c\sqrt{k}}\right)^r$ in (1.6) and making the use of (2.6), we get

$$\gamma_n = \frac{(k;q)_{2n}(q\sqrt{k},-q\sqrt{k},c;q)_n(kq,aq/\sqrt{k},q\sqrt{k}/c,aq/c;q)_\infty}{(kq;q)_{2n}(-\sqrt{k},aq/\sqrt{k},aq/c;q)_n(aq,aq/c\sqrt{k},kq/c,q\sqrt{k};q)_\infty}\left(\frac{aq}{c\sqrt{k}}\right)^n.$$

Substituting $\delta_n\ and\ \gamma_n$ as above in (1.3), we get (2.1).

Proof 2.2 Setting $a = k/a, b = kq^{2n}\ and\ \lambda = q^{2n}\sqrt{ak}$ in (1.11), we get

$$3\phi2(k/a,q^{2n+1}\sqrt{ak},kq^{2n};q^{2n}\sqrt{ak},aq^{2n+1};q,a^2\ /k^2) = \frac{(a/k,a^2q/k;q)_\infty(aq;q)_{2n}}{(a^2/k^2\ ,aq;q)_\infty(a^2\,q/k;q)_{2n}}$$

$$\times\frac{(1-q^{2n}\sqrt{ak}+\sqrt{a}/(\sqrt{k}-a^{3/2}q^{2n}))}{(1-q^{2n}\sqrt{ak})(1+\sqrt{a}/\sqrt{k})},$$

$a^2/k^2| < 1.$

Choosing $\delta r = \frac{(q^{n+1}\sqrt{(ak)};q)_r}{(q^n\sqrt{(ak)};q)_r}\left(\frac{a^2}{k^2}\right)^r$ in (1.6) and substituting in (2.7), we have

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$$\gamma_n = \frac{(1-q^{2n}\sqrt{ak}+\sqrt{a}/(\sqrt{k}-a^{3/2}q^{2n}))}{(1-q^{2n}\sqrt{(ak)})(1+\sqrt{a}/\sqrt{k})}\frac{(a/k,a^2q/k;q)_\infty(k,kq^n,q^{n+1}\sqrt{(ak)};q)_n}{(aq,a^2/k^2;q)_\infty(q^n\sqrt{(ak)},a^2q/k,a^2q^{n+1}/k;q)_n}\left(\frac{a^2}{k^2}\right)^n.$$

using δ_n and γ_n in (1.3), we obtain (2.2).

Applications

By using (1.7) in (2.1) and taking $n \to \infty$, we get

$$8\phi7(k,q\sqrt{k},-q\sqrt{k},c,a,q\sqrt{a},-q\sqrt{a},a/k;\ kq,-\sqrt{k},aq/\sqrt{k}\sqrt{a},-\sqrt{a},kq,aq/c;q,q\sqrt{k}/c)$$
$$=\frac{(q\sqrt{k},kq/c,aq,aq/c\sqrt{k};q)_\infty}{(kq,aq/\sqrt{k},q\sqrt{k}/c,aq/c;q)_\infty}\ .$$

Again by making the use of (1.9) in (2.1) and taking $n \to \infty$, we obtain

$$10\phi9(k,-q\sqrt{k},c,q\sqrt{k},a,q\sqrt{a},-q\sqrt{a},y,z,a^2\ q/kyz;\ kq,-\sqrt{k},aq/\sqrt{k},aq/c,\sqrt{a},-\sqrt{a},aq/y,aq/z,kyz/a;q,q\sqrt{k}/c)$$
$$=\frac{(aq/c\sqrt{k},aq,kq/c,q\sqrt{k};q)\infty}{(kq,aq/\sqrt{k},q\sqrt{k}/c,aq/c;q)\infty}\ 6\phi_5(-q\sqrt{k},c,ky/a,kz/a,k,aq/yz;-\sqrt{k},kq/c,aq/y,aq/z,kyz/a;q,aq/c\sqrt{k}).$$

On using (1.8) in (2.2) and taking $n \to \infty$, we get

$$2\phi1(k,k/a;aq;q,a^2/k^2) = \frac{(a/k,a^2\ q/k;q)_\infty}{(aq,a^2/k^2\ ;q)_\infty}\frac{(1-\sqrt{ak}+\sqrt{a}/(\sqrt{k}-a^{3/2}))}{(1-\sqrt{ak})(1+\sqrt{a}/\sqrt{k})}\ .$$

By making the use of (1.7) in (2.3) and then taking $n \to \infty$, we obtain

$$7\phi6(a,q\sqrt{a},-q\sqrt{a},a/k,k,q\sqrt{k},-q\sqrt{k};\sqrt{a},-\sqrt{a},kq,a^2/k,\sqrt{k},-\sqrt{k};q,a/kq) = \frac{(aq,a^2/k^2q;q)_\infty}{(a^2/k,a/kq;q)_\infty}\ .$$

In (2.3) using (1.9) and then taking $\to \infty$, we get the following transformation

$$9\phi8(k,q\sqrt{k},-q\sqrt{k},a,q\sqrt{a},-q\sqrt{a},y,z,a\ 2\ q/kyz;a^2\ /k,\sqrt{k},-\sqrt{k},\sqrt{a},-\sqrt{a},aq/y,aq/z,kyz/a;q,a/kq)$$
$$=\frac{(aq,a^2\ /k^2\ q;q)_\infty}{(a^2\ /k,a/kq;q)_\infty}\ 6\phi5(q\sqrt{k},-q\sqrt{k},ky/a,kz/a,k,aq/yz;\sqrt{k},-\sqrt{k},aq/y,aq/z,kyz/a;q,a^2k^2q).$$

Again in (2.4) making the use of (1.7) and taking $n \to \infty$, we obtain the following summation

$$12\phi11(k,q\sqrt{k},-q\sqrt{k},\sqrt{a},-\sqrt{a},\sqrt{aq},-\sqrt{aq},\ k^2q/a,a,q\sqrt{a},-q\sqrt{a},a/k;\sqrt{k},-\sqrt{k}\ kq\sqrt{1/a},-kq\sqrt{1/a},k\sqrt{q/a},-k\sqrt{q/a},\sqrt{a},-\sqrt{a},kq,aq,kq;q;\ k^2q/a^2) = \frac{(kq/a,\ k^2\ q/a;q)_\infty}{(kq,\ k^2\ q/a^2\ ;q)_\infty}\ .$$

Now use (1.9) in (2.4) and taking $n \to \infty$,we have

$$14\phi13(k, q\sqrt{k}, -q\sqrt{k}, \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, k^2q/a, a, q\sqrt{a}, -q\sqrt{a}, y, z, a^2q/kyz; \sqrt{k}, -\sqrt{k}, kq\sqrt{1/a}, -kq\sqrt{1/a}, k\sqrt{q/a}, -k\sqrt{q/a}, aq, kq, \sqrt{a}, -\sqrt{a}, aq/y, aq/z, kyz/a; q; k^2q/a^2)$$

$$= \frac{(kq/a,\ k^2q/a; q)_\infty}{(kq,\ k^2q/a^2\ ; q)_\infty}\ 10\phi9(q\sqrt{k}, -q\sqrt{k}, \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, ky/a, kz/a, k, aq/yz; \sqrt{k}, -\sqrt{k}, kq\sqrt{1/a}, -kq\sqrt{1/a}, k\sqrt{q/a}, -k\sqrt{q/a}, aq/y, aq/z, kyz/a; q; kq/a).$$

By using (1.7) in (2.5), we have

$$\sum_{n=0}^{\infty} \frac{(k, kq^n, a, q\sqrt{a}, -q\sqrt{a}, a/k; q)_n}{(q, \sqrt{a}, -\sqrt{a}, kq; q)_n (kq, a^2q^2/k; q^2)_{2n}} (-q)^n = \frac{(-aq/k, aq; q)_\infty}{(-q; q)_\infty (kq, a^2q^2/k; q^2)_\infty}.$$

and again in (2.5) using (1.9), we get

$$4\phi3(ky/a, kz/a, k, aq/yz; aq/y, aq/z, kyz/a; q, -aq/k)$$

$$= \frac{(-q; q)\infty(kq, a\ 2q\ 2/k; q\ 2)\infty}{(-aq/k, aq; q)\infty} \sum_{n=0}^{\infty} \frac{(k, kq^n, a, q\sqrt{a}, -q\sqrt{a}, y, z, a^2q/kyz; q)_n}{(kq, a^2q^2/k; q^2)_{2n} (q, \sqrt{a}, -\sqrt{a}, aq/y, aq/z, kyz/a; q)_n} (-q)^n.$$

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