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CERTAIN TRANSFORMATION AND SUMMATION FORMULAE FOR POLY- BASIC HYPERGEOMETRIC SERIES

Dr. Brijesh Pratap Singh
Assistant Professor
Department of Mathematics
Raja Harpal Singh Mahavidyalaya,
Singramau, Jaunpur (U.P.)

Abstract:
We offer an overview of some of the main findings from the hypergeometric sequence theories and integrals associated with root systems. In particular, for such multiple series and integrals, we list a number of summations, transformations and explicit evaluations. Interesting transformation formulas for poly-basic hypergeometry using some known summation formulae and the identity defined herein. In particular, for such multiple series and integrals, we list a number of summations, transformations and explicit evaluations. Interesting transformation formulas for poly-basic hypergeometric sequence have been constructed using some known summation formulae and the identity set out herein.
Keywords: summation formula, transformation formula, basic hypergeometric series, poly-basic hypergeometric series.

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Corresponding author: Dr. Brijesh Pratap Singh

Introduction:

Due to their applications in various fields, such as additive number theory, combinatorial analysis, statistical and quantum mechanics, vector spaces, etc, simple hypergeometric series have assumed considerable importance over the last four decades or so. They also developed a very useful method for analysts to unify and sub-sum various isolated findings under a common umbrella in the theory of numbers.The enormous mass of literature on basic hypergeometric series has become so important and important (or q-hypergeometric series as we sometimes call it) that their analysis has acquired its own separate, reputable status rather than being viewed merely as a generalization of the ordinary hypergeometric series.

The discovery of Ramanujan's 'Lost' Note book by G.E. Andrews in 1976 aroused a new interest in these functions. He gave a beautiful account of the discovery of the 'Lost' Notebook and its contents in the American Mathematical Monthly in 1979.

W.N. Bailey in 1944, gave the following result :

If

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r}$$

and

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n}$$

where α_r, δ_r, u_r and v_r are any function of r only, such that the series γ_n exists,

then

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n. \tag{1}$$

The above transformation leads to multiple outcomes that play important roles in hypergeometric series number theory and transformation theory. We demonstrate here that this transformation can be used to define some poly-basic hypergeometric series transformations.

If we take $u_r=v_r=1$ and $\delta_r=z^r$ in (3.1.1), we get :

If

$$\beta_n = \sum_{r=0}^{n} \alpha_r \tag{2}$$

then $$\sum_{n=0}^{\infty} \alpha_n z^n = (1-z) \sum_{n=0}^{\infty} \beta_n z^n, \tag{3}$$

because $\gamma_n = \dfrac{z^n}{1-z}, |z|<1.$

We shall use the following known sums of truncated series to derive our transformations.

$$
{}_2\Phi_1\left[\begin{matrix} a, y; q; q \\ ayq \end{matrix}\right]_N = \frac{[aq, yq; q]_N}{[q, ayq; q]_N}. \tag{4}
$$

[Agarwal, R.P. 5; App. II (8)]

$$
{}_4\Phi_3\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, e; q; 1/e \\ \sqrt{a}, -\sqrt{a}, aq/e \end{matrix}\right]_N = \frac{[aq, eq; q]_N}{[q, aq/e; q]_N\, e^N}. \tag{5}
$$

[Agarwal, R.P. 5; App. II (23)]

$$
{}_6\Phi_5\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d \quad ; q; q \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d \end{matrix}\right]_N
$$

$$
= \frac{[aq, bq, cq, dq; q]_N}{[q, aq/b, aq/c, aq/d; q]_N}, \ (a=bcd). \tag{6}
$$

[Agarwal, R.P. 5; App. II (25)]

$$
{}_3\Phi_2\left[\begin{matrix} a, b, q; q; q \\ e, f \end{matrix}\right]_N = \frac{(q-e)(e-abq)}{(aq-e)(e-bq)}\left[1 - \frac{[a, b; q]_{N+1}}{[e/q, abq/e; q]_{N+1}}\right], \tag{7}
$$

where $ef=abq^2$.

[Srivastava, A.K. 3; (4.2)]

$$
\sum_{k=0}^{n} \frac{(1-ap^k q^k)[a; p]_k [c; q]_k c^{-k}}{(1-a)[q; q]_n [ap/c; p]_n}
$$

$$
= \frac{[ap; p]_n [cq; q]_n c^{-n}}{[q; q]_n [ap/c; p]_n}. \tag{8}
$$

[Gasper & Rahman 1; App. II (34)]

$$\sum_{k=0}^{n} \frac{(1-ap^k q^k)(1-bp^k q^{-k})[a,b;p]_k [c,a/bc;q]_k \; q^k}{(1-a)(1-b)[q,aq/b;q]_k \, [ap/c,bcp;p]_k}$$

$$= \frac{[ap,bp;p]_n [cq,aq/bc;q]_n}{[q,aq/b;q]_n \, [ap/c,bcp;p]_n} . \tag{9}$$

[Gasper & Rahman 1; App. II (35)]

$$\sum_{k=0}^{n} \frac{(1-adp^k q^k)(1-\frac{b}{d} p^k q^{-k})[a,b;p]_k [c,ad^2/bc;q]_k \; q^k}{(1-ad)(1-b/d)[dq,adq/b;q]_k \, [adp/c,bcp/d;p]_k}$$

$$= \frac{(1-a)(1-b)(1-c)(1-ad^2/bc)}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)} \times$$

$$\times \frac{[ap,bp;p]_n [cq,ad^2q/bc;q]_n}{[dq,adq/b;q]_n \, [adp/c,bcp/d;p]_n} + \frac{a^2 d(1-d)(1-c/ad)(1-d/bc)(1-b/cd)}{(1-ad)(1-b/d)(1-c/d)(1-ad/bc)} . \tag{10}$$

[Gasper & Rahman 1; App. II (II.36), with m=0]

$$\sum_{k=0}^{n} \frac{(1-adp^k q^k P^k Q^k)(c-dP^k Q^k / p^k q^k)(1-bp^k P^k / dq^k Q^k) \, q^{2k}}{(1-ad)(c-d)(1-b/d)} \times$$

$$\times \frac{(1-adp^k Q^k / bcq^k P^k) q^{2k} [a;p^2]_k [c;q^2]_k [b;P^2]_k [ad^2/bc;Q^2]}{(1-ad/bc)\left[d\frac{qPQ}{p};\frac{qPQ}{p}\right]_k \left[\frac{ad}{c}\frac{pPQ}{q};\frac{pPQ}{q}\right]_k \left[\frac{ad}{b}\frac{pqQ}{P};\frac{pqQ}{P}\right]_k} \times$$

$$\times \frac{1}{\left[\frac{bc}{d} \cdot \frac{pqP}{Q}; \frac{pqP}{Q}\right]_k}$$

$$= \frac{(1-a)(1-b)(1-c)(1-ad^2/bc)}{(1-ad)(c-d)(1-b/d)(1-ad/bc)} \times$$

$$\times \frac{[ap^2;p^2]_n[cq^2;q^2]_n[bP^2;P^2]_n[ad^2Q^2/bc;Q^2]_n}{\left[d\frac{qPQ}{p};\frac{qPQ}{p}\right]_n\left[\frac{ad}{c}\frac{pPQ}{q},\frac{pPQ}{q}\right]_n\left[\frac{ad}{b}\frac{pqQ}{P};\frac{pqQ}{P}\right]_n} \times$$

$$\frac{1}{\left[\frac{bc}{d}\frac{pqP}{Q};\frac{pqP}{Q}\right]_n} + \frac{a^2(1-d)(1-c/ad)(1-b/ad)(1-d/bc)}{(1-ad)(c-d)(1-b/d)(1-ad/bc)}. \qquad (11)$$

[Verma, A. 3; (18) p. 89, with m=0]

$$\sum_{k=0}^{n} \frac{[\beta;p]_k [c;q]_k [y;P]_k [\beta yc/d^2;pP/q]_k\, q^k}{[dq;q]_k [\beta cp/d;p]_k \left[\frac{\beta y}{d},\frac{pP}{q};\frac{pP}{q}\right]_k [cyP/d;P]_k} \times$$

$$\times\left(1-\frac{\beta cy}{d}p^k P^k\right)\left(1-\frac{y}{d}P^k q^{-k}\right)\left(1-\frac{\beta}{d}p^k q^{-k}\right)$$

$$=\frac{(1-d)(1-cy/d)(1-\beta y/d)}{(c-d)}-\frac{(1-\beta)(1-c)(1-y)(1-\beta cy/d^2)}{(c-d)}\times$$

$$\times\frac{[\beta p;p]_n [cq;q]_n [yP;P]_n \left[\frac{\beta cy}{d^2}\frac{pP}{q};\frac{pP}{q}\right]_n}{\left[\frac{\beta cp}{d};p\right]_n [dp;q]_n \left[\frac{cyP}{d};P\right]_n \left[\frac{\beta y}{d}\frac{pP}{q};\frac{pP}{q}\right]_n}. \qquad (12)$$

[Verma 3; 12 (A) with m=0]

Transformation and Summation Formulae :

Here, we shall adopt the following notations and definitions. The q-rising factorial is defined as, for |q| < 1,

$$(a;q)_n = (1-a)(1-aq)...(1-aq^{n-1}), \qquad n \geq 1, \qquad (1.1)$$

$$(a;q)_0 = 1, \qquad (1.2)$$

$$(a;q)_\infty = \prod_{r=0}^{\infty}(1-aq^r) \qquad (1.3)$$

$$(a_1, a_2, a_3, ..., a_r; q)_n = (a_1;q)_n(a_2;q)_n...(a_r;q)_n. \qquad (1.4)$$

A basic hypergeometric series (q-series) is defined by

$$ {}_r\Phi_s\left[\begin{array}{l} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, ..., a_r; q)_n z^n}{(q, b_1, b_2, ..., b_s; q)_n} \left\{(-1)^n q^{n(n-1)/2}\right\}^{1+s-r}, \quad (1.5) $$

[3; (1.2.22) p. 4]

A poly-basic hypergeometric series is defined as,

$$ \Phi\left[\begin{array}{l} a_1, a_2, ..., a_r : c_{1,1}, ..., c_{1,r_1}; ...; c_{m,1}, ...c_{m,r_m}; q, q_1, ..., q_m; z \\ b_1, b_2, ..., b_s : d_{1,1}, ..., d_{1s_1}; ...; d_{m,1}, ..., d_{m,s_m} \end{array}\right] $$

$$ = \sum_{n=0}^{\infty} \frac{(a_1, a_2, ..., a_r; q)_n z^n}{(q, b_1, b_2, ..., b_s; q)_n} \left\{(-1)^n q^{n(n-1)/2}\right\}^{1+s-r} \times $$

$$ \times \prod_{j=1}^{m} \frac{(c_{j,1}, ..., c_{j,r_j}; q_j)_n}{(d_{j,1}, ..., d_{j,s_j}; q_j)_n} \left\{(-1)^n q^{n(n-1)/2}\right\}^{s_j - r_j}. \quad (1.6) $$

[3; (3.9.1), (3.9.2) p. 95]

A truncated basic hypergeometric series defined by

$$ {}_{r+1}\Phi_r\left[\begin{array}{l} a_1, a_2, ..., a_{r+1}; q; z \\ b_1, b_2, ..., b_r \end{array}\right]_N = \sum_{n=0}^{N} \frac{(a_1, a_2, ..., a_{r+1}; q)_n z^n}{(q, b_1, b_2, ..., b_r; q)_n}. \quad (1.7) $$

We shall make use of following summation formulae of truncated basic hypergeometric series in our analysis.

$$ {}_2\Phi_1\left[\begin{array}{l} a, b; q; q \\ abq \end{array}\right]_n = \frac{(aq, bq; q)_n}{(q, abq; q)_n}. \quad (1.8) $$

[Agarwal 1; (2.1) p. 389]

$$ {}_3\Phi_2\left[\begin{array}{l} a, b, q; q; q \\ e, abq^2/e \end{array}\right]_n = \frac{(q-e)(e-abq)}{(aq-e)(e-bq)}\left[1 - \frac{(a, b; q)_{n+1}}{(e/q, abq/e; q)_{n+1}}\right]. \quad (1.9) $$

[Agarwal 2; p. 79]

In the summation formula [Gasper and Rahman 3; App. II (II.21)] if we take $c = aq^{n+1}$, we find the following sum,

$$ {}_4\Phi_3\left[\begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, b; q; 1/b \\ \sqrt{a}, -\sqrt{a}, aq/b \end{array}\right]_n = \frac{(aq, bq; q)_n}{(q, aq/b; q)_n b^n}, \quad |b| > 1. \tag{1.10} $$

Taking $a = bcd$ in [Gasper and Rahman 3; App. II (II. 22)] we find

$$ {}_6\Phi_5\left[\begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, b, c, d; q; q \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d \end{array}\right]_n = \frac{(aq, bq, cq, dq; q)_n}{(q, aq/b, aq/c, aq/d; q)_n}. \tag{1.11} $$

Gasper's indefinite bibasic sum,

$$ \sum_{k=0}^{n} \frac{(1-ap^k q^k)(a;p)_k (c;q)_k}{(1-a)(q;q)_k (ap/c;p)_k} c^{-k} = \frac{(ap;p)_n (cq;q)_n}{(q;q)_n (ap/c;p)_n c^n}. \tag{1.12} $$

[Gasper and Rahman 3; App. II (II. 34)]

We shall also use the following identity,

$$ \sum_{n=0}^{\infty} \alpha_n \sum_{r=0}^{\infty} \delta_r + \sum_{n=0}^{\infty} \alpha_n \delta_n = \sum_{n=0}^{\infty} \alpha_n \sum_{r=0}^{n} \delta_r + \sum_{n=0}^{\infty} \delta_n \sum_{r=0}^{n} \alpha_r \tag{1.13} $$

In Bailey's transform if we take $u_r = v_r = 1$, it takes the following form,

If

$$ \beta_n = \sum_{r=0}^{n} \alpha_r \tag{1.14} $$

and

$$ \gamma_n = \sum_{r=n}^{\infty} \delta_r = \sum_{r=0}^{\infty} \delta_r - \sum_{r=0}^{n} \delta_r + \delta_n \tag{1.15} $$

Then under suitable convergence conditions

$$ \sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n. \tag{1.16} $$

(1.16) can be expressed as,

$$ \sum_{n=0}^{\infty} \alpha_n \left\{ \sum_{r=0}^{\infty} \delta_r - \sum_{r=0}^{n} \delta_r + \delta_n \right\} = \sum_{n=0}^{\infty} \delta_n \sum_{r=0}^{n} \alpha_r, \tag{1.17} $$

which on simplifications gives (1.13)

In this section we shall establish certain transformation formulae for poly-basic Series

series.

(i) Choosing $\alpha_n = \dfrac{(a,b;q)_n q^n}{(q,abq;q)_n}$ and $\delta_n = \dfrac{(\alpha,\beta;p)_n p^n}{(p,\alpha\beta p;p)_n}$ in (1.13) and using (1.8) we get,

$$\frac{(aq,bq;q)_\infty(\alpha p,\beta p;p)_\infty}{(q,abq;q)_\infty(p,\alpha\beta p;p)_\infty} + \Phi\left[\begin{array}{l} a,b:\alpha,\beta;q,p;pq \\ abq:p,\alpha\beta p \end{array}\right]$$

$$= \Phi\left[\begin{array}{l} a,b:\alpha p,\beta p;q,p;q \\ abq:p,\alpha\beta p \end{array}\right] + \Phi\left[\begin{array}{l} aq,bq:\alpha,\beta;q,p;p \\ abq:p,\alpha\beta p \end{array}\right]. \tag{2.1}$$

Result and Discussion:

In this section we shall establish our main results.

(ii) Taking $p=q$ in (2.1) we get,

$$\frac{(aq,bq,\alpha q,\beta q;q)_\infty}{(q,q,abq,\alpha\beta q;q)_\infty} + {}_4\Phi_3\left[\begin{array}{l} a,b,\alpha,\beta;q;q^2 \\ abq,q,\alpha\beta q \end{array}\right]$$

$$= {}_4\Phi_3\left[\begin{array}{l} a,b,\alpha q,\beta q;q;q \\ abq,q,\alpha\beta q \end{array}\right] + {}_4\Phi_3\left[\begin{array}{l} aq,bq,\alpha,\beta;q;q \\ abq,q,\alpha\beta q \end{array}\right]. \tag{2.2}$$

(iii) Taking $\alpha_n = \dfrac{(a,q\sqrt{a},-q\sqrt{a},b;q)_n}{(q,\sqrt{a},-\sqrt{a},aq/b;q)_n b^n}$ and $\delta_n = \dfrac{(\alpha,\beta;p)_n p^n}{(p,\alpha\beta p;p)_n}$ in (1.13) and using (1.10) and (1.8) we obtain,

$$\Phi\left[\begin{array}{l} a,q\sqrt{a},-q\sqrt{a},b:\alpha,\beta;q,p;p/b \\ \sqrt{a},-\sqrt{a},aq/b:p,\alpha\beta p \end{array}\right]$$

$$= \Phi\left[\begin{array}{l} a,q\sqrt{a},-q\sqrt{a},b:\alpha p,\beta p;q,p;1/b \\ \sqrt{a},-\sqrt{a},aq/b:p,\alpha\beta p \end{array}\right] + \Phi\left[\begin{array}{l} aq,bq:\alpha,\beta;q,p;p/b \\ aq/b:p,\alpha\beta p \end{array}\right], \quad \left|\frac{1}{b}\right| < 1. \tag{2.3}$$

For $p=q$, (2.3) yields

$${}_6\Phi_5\left[\begin{array}{l} a,q\sqrt{a},-q\sqrt{a},b,\alpha,\beta;q;q/b \\ q,\sqrt{a},-\sqrt{a},aq/b,\alpha\beta q \end{array}\right]$$

$$= {}_6\Phi_5\left[\begin{array}{l} a,q\sqrt{a},-q\sqrt{a},b,\alpha q,\beta q;q;1/b \\ q,\sqrt{a},-\sqrt{a},aq/b,\alpha\beta q \end{array}\right] + {}_4\Phi_3\left[\begin{array}{l} aq,bq,\alpha,\beta;q;q/b \\ q,aq/b,\alpha\beta q \end{array}\right], \quad |b|>1. \tag{2.4}$$

As $b\to\infty$ in (2.4) we get,

$${}_5\Phi_5\left[\begin{array}{l} a,q\sqrt{a},-q\sqrt{a},\alpha,\beta;q;q \\ q,\sqrt{a},-\sqrt{a},\alpha\beta q,0 \end{array}\right]$$

$$= {}_5\Phi_5\left[\begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, \alpha q, \beta q; q; 1 \\ q, \sqrt{a}, -\sqrt{a}, \alpha\beta q, 0 \end{array}\right] + {}_3\Phi_3\left[\begin{array}{l} aq, \alpha, \beta; q; q^2 \\ q, \alpha\beta q, 0 \end{array}\right]. \tag{2.5}$$

(iv) Taking $\alpha_n = \dfrac{(a, q\sqrt{a}, -q\sqrt{a}, b; q)_n}{(q, \sqrt{a}, -\sqrt{a}, aq/b; q)_n b^n}$ $\quad |b| > 1$,

and $\delta_n = \dfrac{(\alpha, p\sqrt{\alpha}, -p\sqrt{\alpha}, \beta; p)_n}{(p, \sqrt{\alpha}, -\sqrt{\alpha}, \alpha p/\beta; p)_n \beta^n}$, $\quad |\beta| > 1$ in (1.13) and making use of (1.10) we find,

$$\Phi\left[\begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, b : \alpha, p\sqrt{\alpha}, -p\sqrt{\alpha}, \beta; q, p; \dfrac{1}{b\beta} \\ \sqrt{a}, -\sqrt{a}, \dfrac{aq}{b} : p, \sqrt{\alpha}, -\sqrt{\alpha}, \dfrac{\alpha p}{\beta} \end{array}\right]$$

$$= \Phi\left[\begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, b : \alpha p, \beta p; q, p; \dfrac{1}{b\beta} \\ \sqrt{a}, -\sqrt{a}, \dfrac{aq}{b} : p, \dfrac{\alpha p}{\beta} \end{array}\right]$$

$$+\Phi\left[\begin{array}{l} aq, bq : \alpha, p\sqrt{\alpha}, -p\sqrt{\alpha}, \beta; q, p; \dfrac{1}{b\beta} \\ \dfrac{aq}{b} : p, \sqrt{\alpha}, -\sqrt{\alpha}, \dfrac{\alpha p}{\beta} \end{array}\right]. \tag{2.6}$$

For $p = q$, (2.6) gives

$${}_8\Phi_7\left[\begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, b, \alpha, q\sqrt{\alpha}, -q\sqrt{\alpha}, \beta; q; \dfrac{1}{b\beta} \\ q, \sqrt{a}, -\sqrt{a}, \dfrac{aq}{b}, \sqrt{\alpha}, -\sqrt{\alpha}, \dfrac{\alpha q}{\beta} \end{array}\right]$$

$$= {}_6\Phi_5\left[\begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, b, \alpha q, \beta q; q; \dfrac{1}{b\beta} \\ q, \sqrt{a}, -\sqrt{a}, \dfrac{aq}{b}, \dfrac{\alpha q}{\beta} \end{array}\right]$$

$$+ {}_6\Phi_5\left[\begin{array}{l} aq, bq, \alpha, q\sqrt{\alpha}, -q\sqrt{\alpha}, \beta; q; \dfrac{1}{b\beta} \\ q, \dfrac{aq}{b}, \sqrt{\alpha}, -\sqrt{\alpha}, \dfrac{\alpha q}{\beta} \end{array}\right]. \tag{2.7}$$

For $b, \beta \to \infty$, (2.7) yields

For $b, \beta \to \infty$, (2.7) yields

$${}_6\Phi_7\left[\begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, \alpha, q\sqrt{\alpha}, -q\sqrt{\alpha}; q; q \\ q, \sqrt{a}, -\sqrt{a}, \sqrt{\alpha}, -\sqrt{\alpha}, 0, 0 \end{array}\right]$$

$$= {}_4\Phi_5\left[\begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, \alpha q; q; q \\ q, \sqrt{a}, -\sqrt{a}, 0, 0 \end{array}\right] + {}_4\Phi_5\left[\begin{array}{l} aq, \alpha, q\sqrt{\alpha}, -q\sqrt{\alpha}; q; q \\ q, \sqrt{\alpha}, -\sqrt{\alpha}, 0, 0 \end{array}\right]. \tag{2.8}$$

(v) Choosing $\alpha_n = \dfrac{(a,b;q)_n q^n}{(e,abq^2/e;q)_n}$, $\delta_n = \dfrac{(\alpha,\beta;p)_n p^n}{(p,\alpha\beta p;p)_n}$ in (1.13) and using (1.9) and (1.8) we find,

$$\frac{(\alpha p,\beta p;p)_\infty}{(p,\alpha\beta p;p)_\infty}\left\{1-\frac{(a,b;q)_\infty}{(e/q,abq/e;q)_\infty}\right\}\frac{(q-e)(e-abq)}{(aq-e)(e-bq)}+\Phi\left[\begin{array}{l}\alpha,\beta:a,b;p,q;pq\\ \alpha\beta p:e,abq^2/e\end{array}\right]$$

$$=\Phi\left[\begin{array}{l}\alpha p,\beta p:a,b;p,q;q\\ \alpha\beta p:e,abq^2/e\end{array}\right]+\frac{(q-e)(e-abq)}{(aq-e)(e-bq)}\times$$

$$\times\sum_{n=0}^{\infty}\frac{(\alpha,\beta;p)_n p^n}{(p,\alpha\beta p;p)_n}\left\{1-\frac{(a,b;q)_{n+1}}{(e/q,abq/e;q)_{n+1}}\right\}. \qquad (2.9)$$

For $p=q$, (2.9) yields

$${}_4\Phi_3\left[\begin{array}{l}\alpha,\beta,a,b;q;q^2\\ \alpha\beta q,e,abq^2/e\end{array}\right]=\frac{eq(1-a)(1-b)(\alpha q,\beta q,aq,bq;q)_\infty}{(aq-e)(e-bq)(q,\alpha\beta q,e,abq^2/e;q)_\infty}$$

$$+\ {}_4\Phi_3\left[\begin{array}{l}\alpha q,\beta q,a,b;q;q\\ \alpha\beta q,e,abq^2/e\end{array}\right]-\frac{eq(1-a)(1-b)}{(aq-e)(e-bq)}\,{}_4\Phi_3\left[\begin{array}{l}\alpha,\beta,aq,bq;q;q\\ \alpha\beta q,e,abq^2/e\end{array}\right]. \qquad (2.10)$$

If we put $e=q$ in (2.10) we get (2.2) again.

(vi) Taking $\alpha_n = \dfrac{(a,q\sqrt{a},-q\sqrt{a},b,c,d;q)_n q^n}{(q,\sqrt{a},-\sqrt{a},aq/b,aq/c,aq/d;q)_n}$,

and $\delta_n = \dfrac{(\alpha,p\sqrt{\alpha},-p\sqrt{\alpha},\beta,\gamma,\delta;p)_n p^n}{(p,\sqrt{\alpha},-\sqrt{\alpha},\alpha p/\beta,\alpha p/\gamma,\alpha p/\delta;p)_n}$, in (1.13) and making use of (1.11) we have,

$$\frac{(aq,bq,cq,dq;q)_n(\alpha p,\beta p,\gamma p,\delta p;p)_n}{(q,aq/b,aq/c,aq/d;q)_n(p,\alpha p/\beta,\alpha p/\gamma,\alpha p/\delta;p)_n}$$

$$+\Phi\left[\begin{array}{l}a,q\sqrt{a},-q\sqrt{a},b,c,d:\alpha,p\sqrt{\alpha},-p\sqrt{\alpha},\beta,\gamma,\delta;q,p;pq\\ \sqrt{a},-\sqrt{a},aq/b,aq/c,aq/d:p,\sqrt{\alpha},-\sqrt{\alpha},\alpha p/\beta,\alpha p/\gamma,\alpha p/\delta\end{array}\right]$$

$$=\Phi\left[\begin{array}{l}a,q\sqrt{a},-q\sqrt{a},b,c,d:\alpha p,\beta p,\gamma p,\delta p;q,p;q\\ \sqrt{a},-\sqrt{a},aq/b,aq/c,aq/d:p,\alpha p/\beta,\alpha p/\gamma,\alpha p/\delta\end{array}\right]$$

$$+\Phi\left[\begin{array}{l}aq,bq,cq,dq:\alpha,p\sqrt{\alpha},-p\sqrt{\alpha},\beta,\gamma,\delta;q,p;p\\ aq/b,aq/c,aq/d:p,\sqrt{\alpha},-\sqrt{\alpha},\alpha p/\beta,\alpha p/\gamma,\alpha p/\delta\end{array}\right]. \qquad (2.11)$$

For $p=q$, (2.11) yields

$$\frac{(aq,bq,cq,dq,\alpha q,\beta q,\gamma q,\delta q;q)_\infty}{(q,q,aq/b,aq/c,aq/d,\alpha q/\beta,\alpha q/\gamma,\alpha q/\delta;q)_\infty}$$

$$+ {}_{12}\Phi_{11}\left[\begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, \alpha, q\sqrt{\alpha}, -q\sqrt{\alpha}, \beta, \gamma, \delta; q; q^2 \\ q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, \alpha q/\beta, \alpha q/\gamma, \alpha q/\delta \end{array}\right]$$

$$= {}_{10}\Phi_9\left[\begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, \alpha q, \beta q, \gamma q, \delta q; q; q \\ q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, \alpha q/\beta, \alpha q/\gamma, \alpha q/\delta \end{array}\right]$$

$$+ {}_{10}\Phi_9\left[\begin{array}{l} aq, bq, cq, dq, \alpha, q\sqrt{\alpha}, -q\sqrt{\alpha}, \beta, \gamma, \delta; q, q \\ q, aq/b, aq/c, aq/d, \sqrt{\alpha}, -\sqrt{\alpha}, \alpha q/\beta, \alpha q/\gamma, \alpha q/\delta \end{array}\right]. \tag{2.12}$$

If we take $\alpha = a, \beta = b, \gamma = c, \delta = d$ in (2.12) we obtain

$$\left\{\frac{(aq, bq, cq, dq; q)_\infty}{(q, aq/b, aq/c, aq/d; q)_\infty}\right\}^2 + \sum_{n=0}^{\infty}\left\{\frac{(a, q\sqrt{a}, -q\sqrt{a}, b, c, d; q)_n q^n}{(q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d; q)_n}\right\}^2$$

$$+2\ {}_{10}\Phi_9\left[\begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, aq, bq, cq, dq; q; q \\ q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/b, aq/c, aq/d \end{array}\right]. \tag{2.13}$$

Taking $c = a, d = b$ in (2.13) we find,

$$\left\{\frac{(aq, bq; q)_\infty}{(q, aq/b; q)_\infty}\right\}^4 + \sum_{n=0}^{\infty}\left\{\frac{(a, b; q)_n}{(q, aq/b; q)_n}\right\}^4 \left\{\frac{(1-aq^{2n})}{(1-a)}\right\}^2 q^{2n}$$

$$= 2\sum_{n=0}^{\infty}\frac{\{(a, b; q)_{2n}\}^2(1-aq^{2n})q^n}{\{(q, aq/b; q)_n\}^4(1-a)}. \tag{2.14}$$

For $b = a$, (2.14) yields,

$$\left\{\frac{(aq; q)_\infty}{(q; q)_\infty}\right\}^8 + \sum_{n=0}^{\infty}\left\{\frac{(a; q)_n}{(q; q)_n}\right\}^8 \left\{\frac{1-aq^{2n}}{1-a}\right\}^2 q^{2n}$$

$$= 2\sum_{n=0}^{\infty}\frac{(a; q)_{2n}^4}{(q; q)_n^8}\left(\frac{1-aq^{2n}}{1-a}\right)q^n. \tag{2.15}$$

(vii) Choosing $\alpha_r = \dfrac{(1-ap^rq^r)(a; p)_r(c; q)_r}{(1-a)(q; q)_r(ap/c; p)_r c^r}$, $\quad |c| > 1$, and $\delta_n = \dfrac{(\alpha, \beta; q_1)_n q_1^n}{(q_1, \alpha\beta q_1; q_1)_n}$ in (1.13) and using (1.8) and (1.12) we find,

$$\sum_{n=0}^{\infty}\frac{(1-ap^nq^n)(a; p)_n(c; q)_n(\alpha, \beta; q_1)_n}{(1-a)(q; q)_n(ap/c; p)_n(q_1, \alpha\beta q_1; q_1)_n}\left(\frac{q_1}{c}\right)^n$$

(vii) Choosing $\alpha_r = \dfrac{(1-ap^r q^r)(a;p)_r(c;q)_r}{(1-a)(q;q)_r(ap/c;p)_r c^r}$, $|c|>1$, and $\delta_n = \dfrac{(\alpha,\beta;q_1)_n q_1^n}{(q_1,\alpha\beta q_1;q_1)_n}$ in (1.13) and using (1.8) and (1.12) we find,

$$\sum_{n=0}^{\infty} \frac{(1-ap^n q^n)(a;p)_n(c;q)_n(\alpha,\beta;q_1)_n}{(1-a)(q;q)_n(ap/c;p)_n(q_1,\alpha\beta q_1;q_1)_n}\left(\frac{q_1}{c}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{1-ap^n q^n(a;p)_n(c;q)_n}{(1-a)(q;q)_n(ap/c;p)_n}\frac{(\alpha q_1,\beta q_1;q_1)_n}{(q_1,\alpha\beta q_1;q_1)_n}\frac{1}{c^n}$$

$$+\sum_{n=0}^{\infty} \frac{(ap;p)_n(cq;q)_n}{(q;q)_n(ap/c;p)_n}\frac{(\alpha,\beta;q_1)_n}{(q_1,\alpha\beta q_1;q_1)_n}\left(\frac{q_1}{c}\right)^n. \tag{2.16}$$

Taking $p=q_1=q$ in (2.16) we find (2.4).
A number of similar transformations can also be scored.

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