Certain Transformation and Summation Formulae for Poly – Basic Hypergeometric Series

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Abstract

Hypergeometric sequence theories and integrals related to root systems are reviewed in this article. We give a number of summations, transformations, and explicit evaluations for such multiple series and integrals. The identity presented herein and certain well-known summation formulae are used to create some interesting poly-basic hypergeometric transformation formulas. We give a number of summations, transformations, and explicit evaluations for such multiple series and integrals. An attempt has been made to develop transformation formulae involving polybasic hypergeometric series using Bailey's transformation and some known summations of truncated series.

Keyword

Summation formula, transformation formula, basic hypergeometric series, poly - basic hypergeometric series, bailey's pair

Introduction

For the past two centuries, researchers have been studying the theory of hypergeometric series introduced by C. F. Gauss in 1812. Historically, numerical methods have been used to solve mathematical and technical issues from the dawn of human civilization. In engineering, mathematics, and physics, numerical computations are critical in solving real-time and in-the-moment problems. Complex mathematical problems can be solved by applying this strategy of arithmetic operations to them. Mathematical models of physical phenomena that can be solved with mathematical operations are used in this technique. Using computer-oriented numerical approaches is now a standard element of every scientist's day-to-day routine. As ultra-powerful computers have been available, numerical computing methods have become far more powerful, fast, and flexible.

To Bailey's credit, Proceeding of the London Mathematical Society published an article on hypergeometric and fundamental hypergeometric series in 1947.

The following is a crucial finding in the study that was later dubbed Bailley's transformation:

$$if\ \beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r}\ ,$$

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{n+r}$$

$$then \sum_{n=0}^{\infty} \propto_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n\,,$$

where α_r,δ_r,u_r,vr are functions of r only, such that the series for γ_n exists. To further cement the significance of Bailey's transformation was an article he wrote in 1949 for the London Mathematical Society. To prove Rogers-Ramanujan type identities and a wide range of transformations of basic and poly-basic hypergeometric series, the Bailey lemma has proven invaluable. Hypergeometric series theory has seen a lot of attention recently because of its many applications in mathematics and other fields.

We shall make use of following identities

$$\sum_{k=0}^{n} a_k \sum_{j=0}^{n-k} A_j = \sum_{k=0}^{n} A_k \sum_{j=0}^{n-k} a_j \qquad \dots (1)$$

$$and \sum_{m=0}^{n} a_m \sum_{r=0}^{m} A_r = \left[\sum_{r=0}^{n} A_r\right]\left[\sum_{m=0}^{n} a_m\right] - \sum_{r=1}^{n} A_r \sum_{m=0}^{r-1} a_m \qquad \dots (2)$$

because specific poly-basic hypergeometric transformation equations must be established. We'll also need to make use of previously calculated sums. We obtain the following when m = 0 is used in step 2.

$$\sum_{k=0}^{n} \frac{(1-ad\,p^k q^k)\left(1-\frac{b}{d}p^k q^{-k}\right)[a,b;p]_k\left[c,\frac{ad^2}{bc};q\right]_k q^k}{(1-ad)\left(1-\frac{b}{d}\right)\left[dq,\frac{adq}{b};q\right]_k\left[\frac{adp}{c},\frac{bcp}{d};p\right]_k}$$

$$=\frac{d(1-a)(1-b)(1-c)(bc-ad^2)}{(1-ad)(d-b)(d-c)(bc-ad)}$$

$$=\left\{\frac{[ap,bp;p]_n\left[cq,\frac{ad^2}{bc};q\right]}{\left[dq,\frac{adq}{b};q\right]_n\left[\frac{adp}{c},\frac{bcp}{d};p\right]_n}-\frac{(ad-c)(bc-d)(ad-b)(1-d)}{d(1-a)(1-b)(1-c)(ad^2-bc)}\right\}$$

Putting M = 0 in 1 we have

$$\sum_{s=0}^{n} \frac{[\beta;p]_s[c;q]_s\left[\frac{\beta yc}{d^2};\frac{pP}{q}\right]_s}{[dq;q]_s\left[\frac{\beta cp}{d};p\right]_s}$$

$$\times\frac{q^2\left[\left(1-\frac{\beta cy}{d}p^sP^s\right)\left(1-\frac{y}{d}P^s q^{-s}\right)\left(1-\frac{\beta}{d}p^s q^{-s}\right)\right]}{\left(\frac{\beta y}{d}\frac{pP}{q};\frac{pP}{q}\right)\left(\frac{cyP}{d};P\right)_s}$$

$$=\frac{(1-\beta)(1-c)(1-y)\left(1-\frac{\beta cy}{d^2}\right)}{(c-d)}$$

$$\left[\frac{(\beta c-d)(cy-d)(\beta y-d)(1-d)}{(1-\beta)(1-c)(1-y)(\beta cy-d^2)d}-\frac{[\beta p;p]_n[cq;q]_n[yP;P]_n\left[\frac{\beta cy}{d^2}\frac{pP}{q};\frac{pP}{q}\right]}{\left[\frac{\beta cp}{d};p\right]_n[dq;q]_n\left[\frac{cyP}{d};P\right]_n\left[\frac{\beta y}{d}\frac{pP}{q};\frac{pP}{q}\right]_n}\right]$$

The following summation formula is obtained by setting m = 0 in 1.

$$\sum_{k=0}^{n} \frac{(1-ad\,p^k q^k P^k Q^k)\left(c-d\frac{p^k Q^k}{p^k q^k}\right)}{(1-ad)(c-d)}$$

$$\times \frac{\left(1-\frac{b}{d}\frac{p^k P^k}{q^k Q^k}\right)\left(1-\frac{ad}{bc}\frac{P^k q^k}{q^k P^k}\right)}{\left(1-\frac{b}{d}\right)\left(1-\frac{ad}{bc}\right)} q^{2k}$$

$$\times \frac{[a;p^2]_k [c;q^2]_k [b;P^2]_k \left[\frac{ad^2}{bc}Q^2\right]_k}{\left[d\frac{qPQ}{P};\frac{qPQ}{P}\right]_k \left[\frac{ad}{c}\frac{pPQ}{q};\frac{pPQ}{q}\right]_k \left[\frac{ad}{b}\frac{pqQ}{P};\frac{pqQ}{P}\right]_k \left[\frac{bc}{d}\frac{pqP}{Q};\frac{pqP}{Q}\right]_k}$$

$$= \frac{(1-a)(1-b)(1-c)\left(1-\frac{ad^2}{bc}\right)}{(1-ad)(c-d)\left(1-\frac{b}{d}\right)\left(1-\frac{ad}{bc}\right)}$$

$$\times \left[\frac{[ap^2;p^2]_n [cq^2;q^2]_n [bP^2;P^2]_n \left[\frac{ad^2}{bc}Q^2;Q^2\right]}{\left[\left[d\frac{qPQ}{p};\frac{qPQ}{p}\right]_n \left[\frac{ad}{c}\frac{pPQ}{q};\frac{pPQ}{q}\right]_n \left[\frac{ad}{b}\frac{pqQ}{p};\frac{pqQ}{p}\right]_n \left[\frac{bc}{d}\frac{pqP}{Q};\frac{pqP}{Q}\right]_n} - \frac{(1-d)(ad-c)(ad-c)(bc-d)}{(1-a)(1-b)(1-c)(ad-bc)}\right]$$

Review of Literature

Slater, Verma, and Gasper et al. used transformations to derive the following summation identities for fundamental hypergeometric functions:

$$2\Phi_1\begin{bmatrix} a, & b;q & ;c/ab \\ & c & \end{bmatrix} = \frac{(c/a\,,c/b\,;q)\infty}{(c,c/ab\,;q)\infty}$$

$$2\Phi_1\begin{bmatrix} a, & b;q & ;c/ab \\ & c & \end{bmatrix} = \frac{(cq/a\,,cq/b\,;q)\infty}{(cq,cq/ab\,;q)\infty}\left\{\frac{ab(1+c)-c(a+b)}{ab-c}\right\}$$

$$6\Phi_5\begin{bmatrix} a, & q\sqrt{a}, & -q\sqrt{a},b, & c,d; & \\ \sqrt{a}, & -\sqrt{a}, & aq/b\,,aq/c\,,aq/d\,; & & q.aq\,/\,bcd \end{bmatrix} = \frac{(aq,aq/cd\,,aq/bd\,,aq/bc\,;q)\infty}{(aq/b\,,aq/c\,,aq/d\,,aq\,/\,bcd;q)\infty}$$

Here in identities , basic hypergeometric functions summed in ratios of infinite product.

$$3\Phi_2\begin{bmatrix} a, & b,q^{-n} & ;q;q \\ & c, \quad d & \end{bmatrix} = \frac{(c/a\,,c/b\,;q)_n}{(c,c/ab\,;q)_n}$$

$$6\Phi_5\begin{bmatrix} a,q\sqrt{a}, & -q\sqrt{a},b\,,kq^n,q^{-n} & ;q;aq\,/\,bk \\ \sqrt{a},-\sqrt{a},aq/b\,, & aq^{1-n}/k\,,aq^{1+n} & \end{bmatrix} = \frac{(aq,kb\,/\,a;q)_n}{(k/a,aq\,/\,b\,;q)_n b^n}$$

$$8\Phi_7\begin{bmatrix} a,q\sqrt{a}, & -q\sqrt{a},b\,,c,d,a^2q/bck\,,kq^n, & q^{-n}; & \\ \sqrt{a},-\sqrt{a},aq/b\,,aq/c\,,bck/a & aq^{n+1}/k\,,aq^{1+n}; & & q,q \end{bmatrix} = \frac{(aq,aq/bc\,,kb/a\,,kc\,/\,a;q)_n}{(aq/b,aq/c\,,k/a\,,kbc\,/\,a\,;q)_n b^n}$$

Basic hypergeometric functions of identities are summed in q-basic series ratios. This was a thriving topic in combinatorial mathematics in the twentieth century, with several ties to other disciplines. For example, the combinatorics of arranging multiple hyper planes in complex Nspace has novel definitions of hypergeometric series by Aomoto, Israel Gelfand, and others. On the application of hypergeometric functions to fundamental geometry, NJ Fine [Fine et al. (1988)] also worked. Mock theta functions were developed by B.Gorden [Gorden et al. (2000)]. The q-Lagrange inversion formulas were studied by I. Gessel [Gessel et al.(1986)]. M.D. Herschhorn worked on the Rogers-Ramanujan type partition theorem

As Mishra [5] produced conclusions involving one bilateral basic hypergeometric function in terms of another bilateral basic hypergeometric function, a number of mathematicians established fundamental bilateral hypergeometric functions.

$$(a-\alpha q/a)\frac{(\alpha q/ab,-\alpha q/b,-q/b;q)_\infty}{(-\alpha q^2/ab;q)_\infty}2\Psi_2\begin{bmatrix}a,-b;q;\alpha q/ab\\ -\alpha q/b,\alpha q^2/a\end{bmatrix}$$

$$=a(1-\alpha q/ab)\frac{(\alpha q^2/a^2,\alpha q^2/b^2,q^2,\alpha q,q/\alpha;q^2)_\infty}{(\alpha q/a,q/a;q)_\infty}$$

$$-\frac{\alpha q(1-a)(-\alpha q^2,\alpha/ab,-1/b;q)_\infty}{a(-\alpha q/ab;q)_\infty}2\Psi_2\begin{bmatrix}aq.-bq;q;\alpha/ab\\ \alpha q^2/a.\alpha q^2/b\end{bmatrix}$$

$$(a-\alpha q/a)\frac{(-\alpha q/b,\alpha q/ab,-q/b;q)_\infty}{(-\alpha q/ab,\alpha q^2/ab;q)_\infty}2\Psi_2\begin{bmatrix}a,-b;q;\alpha q/ab\\ -\alpha q/b,\alpha q^2/a\end{bmatrix}$$

$$=a\frac{(\alpha q^2/a^2,\alpha q^2/b^2,q^2,\alpha q,q/\alpha;q^2)_\infty}{(\alpha q^2/a,-\alpha q/ab,q/a;q)_\infty}$$

$$-\frac{\alpha q(1-1/a)(-\alpha q^2/b,-\alpha q/a,-q/b-1/b;q)_\infty}{(\alpha q/b,1/a,\alpha q^2/a;q)_\infty}2\Psi_2\begin{bmatrix}-a.-bq;q;-\alpha q/ab\\ -\alpha q^2/b,-\alpha q/a\end{bmatrix}$$

There are some Indian researchers who have made significant contributions to the q function, such as R. P. Agrawal (Agrawal 1967), Agrawal (1976), and Agrawal (1981). For his dissertation, he worked on fractional Q derivatives, q-integrals, and mock theta functions. He also did combinatorial analysis and extended Meijer's G Function, Pade approximants, and continuing fractions. When it came to basic series quadratic transformations, W.A. Al-Salam and A. Verma were the pioneers. Generalized q hypergeometric functions and continuing fractions were among the interests of N. A.Bhagirathi [Bhagirathi (1988)]. Transforms of non-termating basic hypergeometric series, their contour integrals, and applications to Rogers Ramanujan's identities were studied by V. K. Jain and M. Verma [Jain et al, 1980]

Denis and Singh [1] wrote in "Certain Transformations involving Poly Basic Hypergeometric Functions" about poly basic hypergeometric functions being stated in terms of other poly basic hypergeometric functions.

$$3\phi2\begin{bmatrix}c;a;apq;q;p;pq;z\\ -;ap/c;a\end{bmatrix}=(1-cz)2\phi1\begin{bmatrix}cq;ap;p;q;z\\ -;ap/c\end{bmatrix}$$

$$6\phi5\begin{bmatrix}c;a/bc;a,b;apq;bp/q;q,p,pq,p/q;z\\ aq/b,ap/c,bcp;a;b\end{bmatrix}=(1-z)4\phi3\begin{bmatrix}cq,aq/bc,ap,bp;q,p;z\\ aq/b;ap/c,bcp;\end{bmatrix}$$

For truncated basic hypergeometric functions, Yadav and Mishra [10] used a variety of parameters in and in bailey's transform to arrive at the following identities:

$$3\Phi2\begin{bmatrix}a,b,q^{-n}\\ c,abc^{-1}q^{1-n};q,q\end{bmatrix}_n=(1-z)\sum_{n=0}^{\infty}\frac{(a,b,q^{-n};q)_nq^nz^n}{(c,abc^{-1}q^{1-n};q)_n}+\frac{(c/a,c/b;q)_n}{(c,c/ab;q)_n}z^{n+1}$$

$$4\Phi 3\left[\begin{matrix} a, -qa^{1/2}, b, c \\ -a^{1/2}, aq/b, aq/c \end{matrix}; q, qa^{1/2}/bc\right]_n$$

$$= \sum_{n=0}^{\infty} \frac{(a, -qa^{1/2}, b, c; q)_n z^n}{(-a^{1/2}, aq/b, aq/c; q)_n} \left(\frac{qa^{1/2}}{bc}\right)^n$$

$$+ \frac{(aq, q\sqrt{a}/b, q\sqrt{a}/c, aq/bc; q)_\infty}{(aq/b, aq/c, q\sqrt{a}, q\sqrt{a}/bc; q)_\infty} z^{n+1} \qquad (1.12)$$

Research Methodology

In this section we have established the following main results

$$\Phi\left[\begin{matrix} \alpha q, \beta q: a, y; \\ \alpha\beta q: p, ayp; \end{matrix} q, p; p\right]$$

$$= \frac{[ap, yp; p]_\infty}{[p, ayp; p]_\infty} \frac{[\alpha q. \beta q; q]_\infty}{[q, \alpha\beta q; p]_\infty}$$

$$- \frac{q(1-\alpha)(1-\beta)}{(1-q)(1-\alpha\beta q)} \Phi\left[\begin{matrix} ap, yp: \alpha q, \beta q; \\ ayp: q^2, \alpha\beta q^2 \end{matrix} p, q; q\right], \qquad (3.1)$$

$$\Phi\left[\begin{matrix} \alpha q, eq: a, y; \\ \frac{\alpha q}{e}: p, ayp; \end{matrix} q, p; \frac{p}{e}\right]$$

$$= \frac{(1-\alpha q^2)(1-e)}{e(1-q)(1-\alpha q/e)} \times \Phi\left[\begin{matrix} ap, yp: \alpha q, q^2\sqrt{\alpha}, -q^2\sqrt{\alpha}, eq; \\ ayp: q^2, q\sqrt{\alpha}, -q\sqrt{\alpha}, \frac{\alpha q^2}{e}; \end{matrix} p, q; \frac{1}{e}\right], \qquad (3.2)$$

$$\Phi\left[\begin{matrix} \alpha q, \beta q, \gamma q, \delta q: a, y; \\ \frac{\alpha q}{\beta}, \frac{\alpha q}{\gamma}, \frac{\alpha q}{\delta}: p, ayp; \end{matrix} q, p; p\right]$$

$$= \frac{[ap, yp; p]_\infty [\alpha q, \beta q, \gamma q, \delta q; q]_\infty}{[p, ayp; p]_\infty [q, \alpha q/\beta, \alpha q/\gamma, \alpha q/\delta; q]_\infty}$$

$$- \frac{(1-q^2\alpha)(1-\beta)(1-\gamma)(1-\delta)q}{(1-q)(1-\alpha q/\beta)(1-\alpha q/\gamma)(1-\alpha q/\delta)} \qquad (3.3)$$

$$\times \Phi\left[\begin{matrix} ap, yp: \alpha q, q^2\sqrt{\alpha}, -q^2\sqrt{\alpha}, \beta q, \gamma q, \delta q; \\ ayp: q^2, q^2\sqrt{\alpha}, -q\sqrt{\alpha}, \frac{\alpha q^2}{\beta}, \frac{\alpha q^2}{\gamma}, \frac{\alpha q^2}{\delta}; \end{matrix} p, q; q\right],$$

$$\Phi\left[\begin{matrix} x,y,ap,cp; \\ xyP:\frac{\alpha p}{c}:q; \end{matrix} P,p,q;\frac{p}{c}\right]$$

$$=\frac{(1-\alpha pq)(1-c)}{(1-q)(1-\alpha p/c)c} \qquad (3.4)$$

$$\times\Phi\left[\begin{matrix} xP,yP:ap:cq:ap^2q^2; \\ xyP:\frac{ap^2}{c}:q^2:apq; \end{matrix} P,p,q,pq;\frac{1}{c}\right],$$

$$\Phi\left[\begin{matrix} x,y:ap,bp:cq,\frac{\alpha q}{bc}; \\ xyP:\frac{\alpha p}{c},bcp:q,\frac{\alpha q}{b}; \end{matrix} P,p,q;P\right]$$

$$=\frac{[xP,yP;P]_\infty[\alpha p,bp,p]_\infty\,[cq,\alpha q/bc\,;q]_\infty}{[P,xyP;P]_\infty\left[q,\alpha q/b;q\right]_\infty\,[\alpha p/c,bcp\,;p]_\infty}$$

$$-\frac{(1-apq)(1-bp/q)(1-c)(1-\alpha/bc)q}{(1-q)(1-\alpha q/b)(1-\alpha p/c)(1-bcp)} \qquad (3.5)$$

$$\times\Phi\left[\begin{matrix} xP,yP:ap^2q^2,\frac{bp^2}{q^2}:ap,bp:cq,\frac{aq}{bc}; \\ xyP:apq:\frac{bp}{q}:\frac{ap^2}{c},bcp^2:q^2,\frac{aq^2}{b}; \end{matrix} P,pq,\frac{p}{q},p,q;q\right],$$

$$\Phi\left[\begin{matrix} x,y:ap,bp:cq,\frac{ad^2q}{bc}; \\ xyP:\frac{\alpha dp}{c},\frac{bcp}{d}:dq,\frac{adq}{b}; \end{matrix} P,p,q;P\right]$$

$$=\frac{[xP,yP;P]_\infty[ap,bp,p]_\infty\,[cq,ad^2q/bc\,;q]_\infty}{[P,xyP;P]_\infty\left[dq,\alpha dq/b;q\right]_\infty\,[adp/c,bcp\,/\,d;p]_\infty}$$

$$-\frac{dq(1-adpq)(1-bp/dq)(1-c\,/\,d)(1-ad/bc)}{(1-dq)(1-adq/b)(1-\alpha dp/c)(1-bcp\,/\,d)} \qquad (3.6)$$

$$\times\Phi\left[\begin{matrix} xP,yP:adp^2q^2,\frac{bp^2}{dq^2}:ap,bp:cq,\frac{ad^2q}{bc}; \\ xyP:adpq:\frac{bp}{dq}:\frac{adp^2}{c},\frac{bcp^2}{d}:dq^2,\frac{adq^2}{b}; \end{matrix} P,pq,\frac{p}{q},p,q;q\right].$$

Result and Discussion

Taking $u_r = v_r = 1$ in, Bailey's transformation takes the following form:

$$if\, \beta_n = \sum_{r=0}^{n} \alpha_r \,, \qquad (4.1)$$

$$\gamma_n = \sum_{r=0}^{\infty} \delta_r \,, \qquad (4.2)$$

$$then \sum_{n=0}^{\infty} \alpha_n \,\gamma_n = \sum_{r=0}^{\infty} \beta_n \delta_n \,, \qquad (4.3)$$

Proof of Result 3.1

Taking $\alpha r = (\alpha, \beta; q)_r q^r / (q, \alpha\beta q; q)_r$ and $\delta r = (a, y; p)_r p^r/(p, ayp; p)_r$ in 4.1 and 4.2, respectively, we get

$$\beta_n = \frac{(\alpha q, \beta q; q)_n}{(q, \alpha\beta q; q)_n}, \quad \gamma_n = \frac{(ap, yp; p)_\infty}{(p, \alpha yp; p)_\infty} - \frac{(1-ay)(1-p^n)(a, y; p)_n}{(1-a)(1-y)(p, ay; p)_n}\,. \qquad (4.4)$$

Putting these values in (4.3), we get the following transformation:

$$\Phi\begin{bmatrix} \alpha q, \beta q: a, y; \\ \alpha\beta q: p, ayp; \end{bmatrix} q, p; p \Bigg] + \frac{(1-ay)}{(1-a)(1-y)} \Phi\begin{bmatrix} \alpha, \beta: a, y; \\ \alpha\beta q: p, ay; \end{bmatrix} q, p; q \Bigg]$$

$$= \frac{[aP, yP; P]_\infty}{[p, ayp; p]_\infty} \; \frac{[\alpha q, \beta q, q]_\infty}{[q, \alpha\beta q; q]_\infty} + \frac{(1-ay)}{(1-a)(1-y)} \; \Phi\begin{bmatrix} \alpha, \beta: a, y; \\ \alpha\beta q: p, ay; \end{bmatrix} q, p; pq \Bigg] \qquad (4.5)$$

which on simplification gives the result (3.1)

Proof of Result (3.2).

Taking $\alpha_r = \left(\alpha, q\sqrt{\alpha}, -q\sqrt{\alpha}, e; q\right)_r / (q, \sqrt{\alpha}, -\sqrt{\alpha}\,, \alpha q/e\,; q)_r\, e^r$ and d $\delta r = (a,y; p)_r p^r/(p, ayp; p)_r$ in (4.1),(4.2) respectively, and we get

$$\beta_n = \frac{(\alpha q, eq; q)_n}{\left(q, \frac{\alpha q}{e}; q\right)_n e^n},$$

$$\gamma_n = \frac{(ap, yp; p)_\infty}{(p, \alpha yp; p)_\infty} - \frac{(1-ay)(1-p^n)(a, y; p)_n}{(1-a)(1-y)(p, ay; p)_n}\,. \qquad (4.6)$$

Substituting these values in (4.3), we get the following transformation for $|e| > 1$

$$\Phi\begin{bmatrix} \alpha q, eq: a, y; \\ \frac{\alpha q}{e}: p, ayp; \end{bmatrix} q, p; \frac{p}{e} \Bigg]$$

$$= \frac{(1-ay)}{(1-a)(1-e)} \times \Phi\begin{bmatrix} \alpha, q\sqrt{\alpha}, -q\sqrt{\alpha}, e: a, y; \\ \sqrt{\alpha}, -\sqrt{\alpha}, \frac{\alpha q}{e}: p, ay; \end{bmatrix} q, p; \frac{p}{e} \Bigg]$$

$$- \frac{(1-ay)}{(1-a)(1-y)} \Phi\begin{bmatrix} \alpha, q\sqrt{\alpha}, -q\sqrt{\alpha}, e: a, y; \\ \sqrt{\alpha}, -\sqrt{\alpha}, \frac{\alpha q}{e}: p, ay; \end{bmatrix} q, p; \frac{1}{e} \Bigg] \qquad (4.7)$$

which on simplification gives result (3.2)

Proof of Result (3.3)

Taking $\alpha_r = \left(\alpha, q\sqrt{\alpha}, -q\sqrt{\alpha}, \beta, \gamma, \delta; q\right)_r q^r/(q, \sqrt{\alpha}, -\sqrt{\alpha}, \alpha q/\beta, \alpha q/\gamma, \alpha q/\delta; q)_r$, where α = βγδ and δr=(a, y; p)$_r$p^r/(p, ayp; p)$_r$ in (4.1) and (4.2), respectively

$$\beta_n = \frac{(\alpha q, \beta q, \gamma q, \delta q; q)_n}{(q, \alpha q/\beta, \alpha q/\gamma, \alpha q/\delta; q)_n}, \tag{4.8}$$

$$\gamma_n = \frac{(ap, yp; p)_\infty}{(p, \alpha yp; p)_\infty} - \frac{(1-ay)(1-p^n)(a, y; p)_n}{(1-a)(1-y)(p, ay; p)_n}.$$

Substituting these values in (4.3), we get the following transformation for α = βγδ:

$$\Phi\begin{bmatrix} \alpha q, \beta q, \gamma q, \delta q: a, y; \\ \frac{\alpha q}{\beta}, \frac{\alpha q}{\gamma}, \frac{\alpha q}{\delta}: p, ayp; \end{bmatrix} q, p; p \Bigg] + \frac{(1-ay)}{(1-a)(1-y)} \Phi\begin{bmatrix} \alpha, q\sqrt{\alpha}, -q\sqrt{\alpha}, \beta, \gamma, \delta: a, y; \\ \sqrt{\alpha}, -\sqrt{\alpha}, \frac{\alpha q}{\beta}, \frac{\alpha q}{\gamma}, \frac{\alpha q}{\delta}: p, ay; \end{bmatrix} q, p; q \Bigg]$$

$$= \frac{(ap, yp; p)_\infty}{(p, \alpha yp; p)_\infty} \times \frac{(\alpha q, \beta q, \gamma q, \delta q; q)_\infty}{(q, \alpha q/\beta, \alpha q/\gamma, \alpha q/\delta; q)_\infty} + \frac{(1-ay)}{(1-a)(1-y)}$$

$$\times \Phi\begin{bmatrix} \alpha, q\sqrt{\alpha}, -q\sqrt{\alpha}, \beta, \gamma, \delta: a, y; \\ \sqrt{\alpha}, -\sqrt{\alpha}, \frac{\alpha q}{\beta}, \frac{\alpha q}{\gamma}, \frac{\alpha q}{\delta}: p, ay; \end{bmatrix} q, p; pq \Bigg], \tag{4.9}$$

which on simplification gives result (3.3)

Proof of Result (3.4)

Taking $\alpha_r = (apq; pq)_r (a; p)_r (c; q)_r c^{-r}/((a; pq)_r (q; q)_r (ap/c; p)_r)$ and δr =(x, y; P)$_r$P^r/(P, xyP; P)$_r$ in (4.1) and (4.2), respectively and we get

$$\beta_n = \frac{(\alpha p; p)_n (cq; q)_n c^{-n}}{(q; q)_n (ap/c; p)_n},$$

$$\gamma_n = \frac{(xP, yP; P)_\infty}{(P, xyP; P)_\infty} - \frac{(1-xy)(1-P^n)(x, y; P)_n}{(1-x)(1-y)(P, xy; P)_n}. \tag{4.10}$$

Putting these values in (4.3), we get the following transformation for |c| > 1

$$\Phi\begin{bmatrix} x, y: ap: cq; \\ xyP: \frac{\alpha p}{c}: q; \end{bmatrix} P, p, q; \frac{P}{c} \Bigg]$$

$$= \frac{(1-xy)}{(1-x)(1-y)} \times \Phi\begin{bmatrix} x, y: apq: a: c; \\ xy: a: \frac{ap}{c}: q; \end{bmatrix} P, pq, p, q; \frac{P}{c} \Bigg]$$

$$- \frac{(1-xy)}{(1-x)(1-y)} \times \Phi\begin{bmatrix} x, y: apq: a: c; \\ xy: a: \frac{ap}{c}: q; \end{bmatrix} P, pq, p, q; \frac{1}{c} \Bigg], \tag{4.11}$$

which on simplification gives result (3.4)

Conclusion

In the preceding section, we showed how the Bailey lemma can be used to uncover novel summations and transformations of fundamental hypergeometric series. Some of the previous section's transformations generalise well-known formulae for transformation. Research into summation and transformation of basic hypergeometric series can be carried out indefinitely.

Reference

1. Gasper, G. and Rahman M., Basic Hypergeometric Series, Encyclopaedia of Mathematics and Its Applications, Cambridge University Press, New York, USA (1990).
2. Mishra, B. P., On Certain transformation formulae involving Basic Hypergeometric Function, Journal of Ramanujan Society of Mathematics and Mathematical Sciences, Vol. 2, No.2, (2014), 09-16.
3. Slater, L. J., Generalised Hypergeometric functions, Cambridge University Press (1996). Verma, A., On identity of Rogers-Ramanujan type, Indian J. of Pure and Appl. Math. 11(6); 770-790,(1980).
4. Yadav, R. K. and Mishra P. K., Some new transformations on basic hypergeometric series using Bailey's transform, Advances in Applied Science and Research, 5(6), (2014), 97-102.
5. Denis R.Y., Singh S.N. and Singh S.P., Certain Transformation and Summation Formulae for q-basic series, Italian Journal of pure and Applied Mathematics-N, 27, 2010, 179-190
6. R. P. Agarwal, H. L. Manocha and K. Srivastava Rao, Selected topics in special Functions, (Poly basic hypoer- geometric series by A. Verma, (P. 77-92), Allied publishers limited, New Delhi, 2001
7. D. M. Bressoud, M. E. H. Ismail, and D. Stanton, "Change of base in Bailey pairs," The Ramanujan Journal, vol. 4, no. 4, pp. 435–453, 2000.
8. Agarwal, P., A Study of New Trends and Analysis of Special Funtion, LAP Lambert Academic Publishing, 2013.
9. R. Y. Denis, S. N. Singh, and S. P. Singh, "On certain transformation formulae for abnormal q-series,"
10. South East Asian Journal of Mathematics and Mathematical Sciences, vol. 1, no. 3, pp. 7–19, 2003.
11. Andrews, G.E., q-Hypergeometric and Related funcitons (Chapter 17).In NIST Handbook of Mathematical Functions, F.W.J. Olver, D.W. Lozier,R.F. Boisvert, C.W. Clark (eds.), Cambridge University Press, 2010.
12. C. M. Joshi and Y. Vyas, "Extensions of Bailey's transform and applications," International Journal of Mathematics and Mathematical Sciences, no. 12, pp. 1909–1923, 2005.
13. Denis, R.Y., Singh, S.N., Singh, S.P., On certain Transformation and Summation formulae for q-series, Italian Jornal of Pune and Applied Mathematics, 27 (2010), (79-190.
    P. Srivastava, "A note on Bailey's transform," South East Asian Journal of Mathematics and Mathematical Sciences, vol. 2, no. 2, pp. 9–14, 2004.
14. Singh, S., On certain summation and transformation formulas for Basic Hypergeometric series, International Jornal of Mathematics Archive, 2 (12) (2011), 2670-2677.
15. Tariboon, J., Ntouyas, S.K., Agarwal, P., New concepts of fractional quantum calculus and applications to impulsive fractional q-difference equations, Advances in Difference Equations, (2015), 18.
16. Mishra, B. P., On Certain transformation formulae involving Basic Hypergeometric Function, Journal of Ramanujan Society of Mathematics and Mathematical Sciences, Vol. 2, No.2, (2014), 09-16.
17. Laughlin, J. M., Some identities between basic hypergeometric series deriving from a new Bailey type transformation, The J. of Mathematical Analysis and its Applications, 345(2), (2008), 670-677.
18. Laughlin, J. M., and Zimmer, P., Some applications of bailey type transformations, International Mathematical Forum, 5 (2010), 61, 3007-3022.