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CERTAIN TRANSFORMATION AND SUMMATION OF BASIC HYPERGEOMETRIC SERIES

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Abstract

We offer an overview of some of the main findings from the hypergeometric sequence theories and integrals associated with root systems. In particular, for such multiple series and integrals, we list a number of summations, transformations and explicit evaluations. Interesting transformation formulas for poly - basic hypergeometric using some known summation formulae and the identity defined herein. In particular, for such multiple series and integrals, we list a number of summations, transformations and explicit evaluations. Interesting transformation formulas for poly - basic hypergeometric sequence have been constructed using some known summation formulae and the identity set out herein.

Keyword

Summation ,Hypergeometric , Transformation,

Introduction

Hypergeometric series associated with root systems first appeared implicitly in the 1972 work of Aliˇsauskas, Jucys and Jucys and Chac´on, Ciftan and Biedenharn in the con-text of the representation theory of the unitary groups, more precisely, as the multiplicity-free Wigner and Racah coefficients (3jand 6j-symbols) of the group SU(n+ 1). A few years later, Holman, Biedenharn and Louck investigated these coefficients more explicitly as generalized hypergeometric series and obtained a first summation theorem for these.[1] The series in question have explicit summands and contain the Weyl denominator of the root system An, and can thus be considered as hypergeometric series associated with this root system.

In mathematics, basic hypergeometric series, or q-hypergeometric series, are q-analogue generalizations of generalized hypergeometric series, and are in turn generalized by elliptic hypergeometric series. A series x_n is called hypergeometric if the ratio of successive terms x_{n+1}/x_n is a rational function of n. If the ratio of successive terms is a rational function of q_n, then the series is called a basic hypergeometric series. The number q is called the base.[2]

The basic hypergeometric series $2\phi 1(q^{\alpha},q^{\beta};q^{\gamma};q,x)$ was first considered by Eduard Heine (1846). It becomes the hypergeometric series $F(\alpha,\beta;\gamma;x)$ in the limit when base $q=1$

There are two forms of basic hypergeometric series, the unilateral basic hypergeometric series φ, and the more general bilateral basic hypergeometric series ψ. The unilateral basic hypergeometric series is defined as[3]

$$j\phi k\begin{bmatrix} a_1 & a_2 & \dots & a_j \\ b_1 & b_2 & \dots & b_k \end{bmatrix}; q, z\Bigg] = \sum_{n=0}^{\infty} \frac{(a_1,a_2,\dots,a_j;q)_n}{(b_1,b_2,\dots,b_k,q;q)_n}\left((-1)^n q^{\binom{n}{2}}\right)^{1+k-j} z^n$$

Where

$$(a_1,a_2,\dots,a_m;q)_n = (a_1;q)_n(a_2;q)_n\dots(a_m;q)_n$$

And

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) = (1-a)(1-aq)(1-aq^2)\dots(1-aq^{n-1})$$

is the q-shifted factorial. The most important special case is when j = k + 1, when it becomes

$$k+1\phi k\begin{bmatrix} a_1 & a_2 & \cdots & a_k \\ b_1 & b_2 & \dots & b_k \end{bmatrix}; q, z\Bigg] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_{k+1}; q)_n}{(b_1, b_2, \dots, b_k, q; q)_n} z^n$$

This series is called balanced if $a_1 \dots a_{k+1} = b_1 \dots b_k q$. This series is called well poised if $a_1 q = a_2 b_1 = \dots = a_{k+1} b_k$, and very well poised if in addition $a_2 = -a_3 = qa_1 1/2$. The unilateral basic hypergeometric series is a q-analog of the hypergeometric series since

$$\lim_{q\to 1} j\phi k\begin{bmatrix} q^{a_1} & q^{a_2} & \cdots & q^{a_j} \\ q^{b_1} & q^{b_2} & \dots & q^{b_k} \end{bmatrix}; q(q-1)^{1+k-j}, z\Bigg] = jFk\begin{bmatrix} a_1 & a_2 & \cdots & a_j \\ b_1 & b_2 & \dots & b_k \end{bmatrix}; z\Bigg]$$

The bilateral basic hypergeometric series, corresponding to the bilateral hypergeometric series, is defined as[4]

$$j\psi k\begin{bmatrix} a_1 & a_2 & \cdots & a_j \\ b_1 & b_2 & \dots & b_k \end{bmatrix}; q, z\Bigg] = \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \dots, a_j; q)_n}{(b_1, b_2, \dots, b_k; q)_n} \left((-1)^n q^{\binom{n}{2}}\right)^{k-j} z^n$$

The most important special case is when j = k, when it becomes

$$k\psi k\begin{bmatrix} a_1 & a_2 & \cdots & a_k \\ b_1 & b_2 & \dots & b_k \end{bmatrix}; q, z\Bigg] = \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \dots, a_k; q)_n}{(b_1, b_2, \dots, b_k; q)_n} z^n$$

The unilateral series can be obtained as a special case of the bilateral one by setting one of the b variables equal to q, at least when none of the a variables is a power of q, as all the terms with n < 0 then vanish.[5]

The summation formulae for hypergeometric series form a very interesting and useful component of the theory of (basic) hypergeometric series. The -binomial theorem of Cauchy [6] is perhaps the first identity in the class of the summation formulae, which can be stated as

$$\sum_{k=0}^{\infty} \frac{(a)_k}{(q)_k} z^k = \frac{(az)_\infty}{(z)_\infty}, |z| < 1, |q| < 1, \qquad (1.1)$$

Where

$$(a)_\infty := (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k),$$

$$(a)_k := (a;q)_k := \frac{(a)_\infty}{(aq^k)_\infty}, \quad k \text{ is an integer}.$$

For more details about the q-binomial theorem and about the identities which fall in this sequel, one may refer to the book by Gasper and Rahman. Another famous identity in the sequel is the Ramanujan's ${}_1\psi_1$ summation formula.

$$\sum_{k=-\infty}^{\infty} \frac{(a)_k}{(b)_k} z^k = \frac{(az)_\infty (q)_\infty (q/az)_\infty (b/a)_\infty}{(z)_\infty (b)_\infty (b/az)_\infty (q/a)_\infty}, \left|\frac{b}{a}\right| < |z| < 1, \qquad |q| < 1.$$

The transformation formula involving the generalized hypergeometric function to be established in this paper is[7]

$$\sum_{n=0}^{\infty}\frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\frac{(d)_n}{n!}x^n y^n \quad 2F2\begin{bmatrix}d+n,e+1; & \\ f+1,e; & x\end{bmatrix}$$

$$=\sum_{n=0}^{\infty}\frac{(d)_n\dots(e+1)_n}{(f+1)_n\dots(e)_n}\frac{x^n}{n!}\times p+3Fq$$

$$+1\begin{bmatrix}-n,1-e-n,-f-n,a_1,\dots a_p; & \\ -e-n,b_1,\dots b_q & y\end{bmatrix}.$$

Proof In order to prove, we proceed as follows. If we denote the left-hand side of above equation by S and express ${}_2F_2$ as a series, we have[8]

$$S=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\cdot\frac{(d)_n(d+n)_m(e+1)_m}{(f+1)_m\dots(e)_m\,n!\,m!}x^{n+m}y^n,$$

which, upon use of the identity

$$(d)_n(d+n)_m=(d)_{m+n},$$

Becomes

$$S=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\cdot\frac{(d)_{n+m}(e+1)_m}{(f+1)_m\dots(e)_m\,n!\,m!}x^{n+m}y^n,$$

Now replacing m by m−n and making use of a simple manipulation for the double series

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k),$$

we have

$$S=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\cdot\frac{(d)_m(e+1)_{m-n}}{(f+1)_{m-n}\dots(e)_{m-n}\,n!\,m!}\frac{x^m y^n}{n!\,(m-n)!}.$$

Using the series identities

$$(a)_{m-n}=\frac{(-1)^n(a)_m}{(1-a-m)_n}$$

$$(m-n)!=\frac{(-1)^n m!}{(-m)_n}$$

in the last summation, after little simplifications, we get

$$S=\sum_{m=0}^{\infty}\frac{(d)_m(e+1)_m}{(f+1)_m(e)_m m}\frac{x^m}{m!}\sum_{n=0}^{m}\frac{(a_1)_n\dots(a_p)_n(-f-m)_n(1-e-m)_n(-m)_n}{(b_1)_n\dots(b_q)_n(-e-m)_n n!}y^n$$

Finally, expressing the inner series as $a_{p+3}F_{q+2}(y)$hypergeometric function, we then easily arrive at the right-hand side[9]

The summation formulas to be established in this section are given by the following theorems.

Theorem 1 The following formula holds true:[10]

$$e^{-x}\sum_{n=0}^{\infty}\frac{x^n}{n!}2F2\left[\begin{matrix}-n,e+1;\\ v+2,e;\end{matrix}x\right]$$
$$=2F3\left[\begin{matrix}\frac{1}{2}v+\frac{1}{2},\frac{1}{2}e+1;\\ v+1,\frac{1}{2}e,\frac{1}{2}v+\frac{3}{2};\end{matrix}-x^2\right]$$
$$+\frac{(e-v-1)x}{e(v+2)}1F2\left[\begin{matrix}\frac{1}{2}v+1;\\ v+2,\frac{1}{2}v+2\end{matrix}-x^2\right].$$

Theorem 2 The following formula holds true:

$$e^{-x}\sum_{n=0}^{\infty}\frac{(1+v)_n}{(1-v)_n}\frac{x^n}{n!}2F2\left[\begin{matrix}-n,e+1;\\ v+2,e;\end{matrix}x\right]=\frac{\Gamma(\frac{1}{2})\Gamma(1-v)}{e2^{-v-1}}\left\{\frac{(e-v-1)}{\Gamma(\frac{1}{2}v)\Gamma(\frac{1}{2}-\frac{1}{2}v)}2F3\left[\begin{matrix}\frac{1}{2}v+\frac{1}{2},\frac{1}{2}v+1;\\ \frac{3}{2},\frac{1}{2}v+\frac{3}{2},\frac{1}{2}-\frac{1}{2}v;\end{matrix}-\right.\right.$$

*x2−e−v**Γ−12v−12Γ1−12v2F312v+12,12e−12v+1;32,12e−12v,1−12v;−x2−xv+1 v+2.Γ12Γ1−ve2−v−1e−v−1Γ−12v−12Γ1−12v×3F41,12v+1, 12v+32;2,32,12v+2, 1−12v;−x2−1+e−v**Γ−12v−1Γ32−12v×3F41,12v+1,32+ 12e−12v;2,32,12+12e−12v, 32−12v;−x2 .*

Theorem 3 The following formula holds true:[11-12]

$$e^{-x}\sum_{n=0}^{\infty}\frac{(1+v)_n}{(\mu)_n}.\frac{(-x)^n}{n!}2F2\left[\begin{matrix}-n,e+1;\\ v+2,e;\end{matrix}x\right]$$
$$=5F5\left[\begin{matrix}v+1,\frac{1}{2}\mu+\frac{1}{2}v,\frac{1}{2}\mu+\frac{1}{2}v+\frac{1}{2},e+\frac{1}{2}\mu-\frac{1}{2}A+\frac{1}{2},e+\frac{1}{2}\mu+\frac{1}{2}A+\frac{1}{2};\\ v+2,\mu,\mu+v+1,e+\frac{1}{2}\mu-\frac{1}{2}A-\frac{1}{2},e+\frac{1}{2}\mu+\frac{1}{2}A-\frac{1}{2};\end{matrix}-4x\right]$$
$$A^2=(\mu-1+2e)^2-4e(\mu+v)\,.$$

Research Methodology

A research methodology is a universal way to addressing a study subject through data collection, data evaluation, and results based on the findings of the study. A research technique is a plan for carrying out a research study. The methodical gathering and analysis of facts and information for the advancement of knowledge in any area may be loosely defined as research. The goal of the study is to use systematic techniques to find solutions to intellectual and practical problems.

Result and Discussion

In order to establish Theorem 1, we proceed as follows.

$$e^x\sum_{n=0}^{\infty}\frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\frac{(d)_n}{n!}x^ny^n 2F2\left[\begin{matrix}f-d-n,e+1;\\ f+1,e;\end{matrix}-x\right]=\sum_{n=0}^{\infty}\frac{(d)_n(e+1)_n}{(f+1)_n\dots(e)_n}\frac{x^n}{n!}p + 3Fq+1\left[\begin{matrix}-n,1-e-n,-f-n,a_1,\dots,a_p;\\ -e-n,b_1,\dots,b_p;\end{matrix}y\right]$$

Putting d=f in above, we get

$$e^x\sum_{n=0}^{\infty}\frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\frac{(f)_n}{n!}x^ny^n 2F2\left[\begin{matrix}-n,e+1;\\ f+1,e;\end{matrix}-x\right]=\sum_{n=0}^{\infty}\frac{(f)_n(e+1)_n}{(f+1)_n(e)_n}\frac{x^n}{n!}p+3Fq + 1\left[\begin{matrix}-n,1-e-n,-f-n,a_1,\dots,a_p;\\ -e-n,b_1,\dots,b_p;\end{matrix}y\right]$$

Now, replacing x by −x and putting f=v+1, we have[13]

$$e^x\sum_{n=0}^{\infty}\frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\frac{(v+1)_n}{n!}(-x)^ny^n 2F2\left[\begin{matrix}-n,e+1;\\ v+2,e;\end{matrix}-x\right]=\sum_{n=0}^{\infty}\frac{(v+1)_n(e+1)_n}{(v+2)_n(e)_n}\frac{x^n}{n!}p+3Fq + 1\left[\begin{matrix}-n,1-e-n,-v-1-n,a_1,\dots,a_p;\\ -e-n,b_1,\dots,b_q;\end{matrix}y\right]$$

Now, set p=0, q=1, b1=v+1, and y=−1 in above, and we get

$$e^{-x}\sum_{n=0}^{\infty}\frac{x^n}{n!}2F2\left[\begin{matrix}-n,e+1;\\ v+2,e;\end{matrix}x\right]=\sum_{n=0}^{\infty}\frac{(v+1)_n(e+1)_n}{(v+2)_n(e)_n}\frac{(-x)^n}{n!}\;3F2\left[\begin{matrix}-n,-v-1-n,1-e-n;\\ v+1,-e-n;\end{matrix}-1\right],$$

which is valid for |x|<∞.

Now, it is readily seen that the $_3F_2$ on the right-hand side of above can be evaluated with the help of the contiguous extension of Kummer's summation theorem[14]

$$e^{-x}\sum_{n=0}^{\infty}\frac{x^n}{n!}2F2\left[\begin{matrix}-n,e+1;\\ v+2,e;\end{matrix}x\right]=\sum_{n=0}^{\infty}\frac{(v+1)_n(e+1)_n}{(v+2)_n(e)_n}\frac{(-x)^n}{n!}\left[\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(v+1)}{2^{-n}\Gamma\left(-\frac{1}{2}n\right)\Gamma\left(\frac{1}{2}n+v+\frac{3}{2}\right)}\left(1-\frac{v+1+n}{e+n}\right)+\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(v+1)}{2^{-n}\Gamma\left(-\frac{1}{2}n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}n+v+1\right)}\right],$$

Now, separating the terms appearing on the right-hand side of above into even and odd powers of x and making use of the following elementary identities:.

$$\Gamma(a-n)=\frac{(-1)^n\Gamma(a)}{(1-n)_n},\quad 2n!=2^{2n}n!\left(\frac{1}{2}\right)_n,$$

$$(2n+1)! = 2^{2n} n! \left(\frac{3}{2}\right), \quad (a)_{2n} = 2^{2n}\left(\frac{1}{2}a\right)_n \left(\frac{1}{2}a+\frac{1}{2}\right)_n$$

$$(a)_{2n+1} = a2^{2n}\left(\frac{1}{2}a+\frac{1}{2}\right)_n \left(\frac{1}{2}a+1\right)_n$$

and after some straightforward calculation, we have[15]

$$e^{-x}\sum_{n=0}^{\infty}\frac{x^n}{n!}2F2\left[\begin{matrix}-n, e+1; \\ v+2, e;\end{matrix} x\right] = \sum_{n=0}^{\infty}\frac{(\frac{1}{2}v+\frac{1}{2})_n(\frac{1}{2}e+1)_n}{(v+1)_n(\frac{1}{2}e)_n(\frac{1}{2}v+\frac{3}{2})_n}\frac{(-1)^n x^{2n}}{n!}$$

$$+\frac{(e-1-v)x}{e(v+2)}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}v+\frac{1}{2}\right)_n(-1)^n}{(v+2)_n(\frac{1}{2}+2)_n}\frac{x^{2n}}{n!}.$$

Finally, summing up the two series on the right-hand side, we easily arrive at the desired result

In order to establish Theorem 2, if we set p=0, q=1, b1=1−ν and y=−1 we have

$$e^{-x}\sum_{n=0}^{\infty}\frac{(v+1)_n}{(1-v)_n}\frac{x^n}{n!}2F2\left[\begin{matrix}-n, e+1; \\ v+2, e;\end{matrix} x\right]$$

$$=\sum_{n=0}^{\infty}\frac{(v+1)_n(e+1)_n}{(v+2)_n(e)_n}\frac{(-x)^n}{n!}3F2\left[\begin{matrix}-n, -v-1-n, 1-e-n; \\ 1-v, e-n;\end{matrix} -1\right]$$

which is valid for $|x|<\infty$.

It is now easy to see that the $_3F_2$ on the right-hand side of above can be evaluated with the help of the extension of Kummer's summation theorem ; we get

$$e^{-x}\sum_{n=0}^{\infty}\frac{(v+1)_n}{(1-v)_n}\frac{x^n}{n!}2F2\left[\begin{matrix}-n, e+1; \\ v+2, e;\end{matrix} x\right]$$

$$=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(1-v)}{2^{-v-1}}\sum_{n=0}^{\infty}\frac{(v+1)_n(e+1)_n(-x)^n}{(v+2)_n(e)_n(e+n)2^{-n}\Gamma(n+2)}$$

$$\times\left\{\frac{(e-v-1)}{\Gamma\left(-\frac{1}{2}n-\frac{1}{2}v\right)\Gamma\left(\frac{1}{2}n-\frac{1}{2}v+\frac{1}{2}\right)} - \frac{(e-v-n)}{\Gamma\left(-\frac{1}{2}n-\frac{1}{2}v-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}n-\frac{1}{2}v+1\right)}\right\}.$$

Now, separating the terms appearing on the right-hand side of above into even and odd powers of x and making use of the elementary identities given in and after some straightforward algebra, we get

$$e^{-x}\sum_{n=0}^{\infty}\frac{(1+v)_n}{(1-v)_n}\frac{x^n}{n!}2F2\left[\begin{matrix}-n,e+1;\\ v+2,e;\end{matrix}x\right]$$

$$=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(1-v)}{e2^{-v-1}}\left\{\frac{(e-v-1)}{\Gamma\left(-\frac{1}{2}v\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}v\right)}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}v+\frac{1}{2}\right)_n\left(\frac{1}{2}v+1\right)_n(-1)^n x^{2n}}{\left(\frac{3}{2}\right)_n\left(\frac{1}{2}e-\frac{1}{2}v\right)_n\left(1-\frac{1}{2}v\right)_n n!}\right\}$$

$$-x\frac{(v+1)}{(v+2)}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(1-v)}{e2^{-v-1}}\left\{\frac{(e-v-1)}{\Gamma\left(-\frac{1}{2}v-\frac{1}{2}\right)\Gamma\left(1-\frac{1}{2}v\right)}\right.$$

$$\times\sum_{n=0}^{\infty}\frac{(1)_n\left(\frac{1}{2}v+1\right)_n\left(\frac{1}{2}v+\frac{3}{2}\right)_n(-1)^n x^{2n}}{(2)_n\left(\frac{3}{2}\right)_n\left(\frac{1}{2}v+2\right)_n\left(1-\frac{1}{2}v\right)_n n!}$$

$$\left.-\frac{(1-e-v)}{\Gamma\left(-\frac{1}{2}v+1\right)\Gamma\left(\frac{3}{2}-\frac{1}{2}v\right)}\sum_{n=0}^{\infty}\frac{(1)_n\left(\frac{1}{2}v+1\right)_n\left(\frac{3}{2}+\frac{1}{2}e-\frac{1}{2}v\right)_n(-1)^n x^{2n}}{(2)_n\left(\frac{3}{2}\right)_n\left(\frac{3}{2}+\frac{1}{2}e-\frac{1}{2}v\right)_n\left(\frac{3}{2}-\frac{1}{2}v\right)_n n!}\right\}$$

Finally, summing up the four series on the right-hand side, we arrive at the desired result

Conclusion

Results founds in the section are very useful and interesting summations formulae in the light of basic hypergeometric functions by make use of Bailey's transform and some different summations formulae.

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