A STUDY ON HYPERGEOMETRIC FUNCTION

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Abstract
A generalisation of exponential functions, hypergeometric functions are referred to as. These are clearly defined functions that can be manipulated analytically as well as numerically. In quantitative economics, all of the functions and series we utilise are subsets of them. Hypergeometric functions 1F1(a; b; z; c) and 2F1(a, b; c; z) for distinct parameter regimes within the complex plane for parameters a and b for 1F1 and b and c for 2F1, as well as different regimes of the complex variable z in both cases. For this, we use a variety of algorithms and approaches, including Taylor and asymptotic series calculations, quadrature, and numerical solution of differential equations and recurrence relations, amongst other ideas.

Keyword
Hypergeometric, Applied mathematics, hypergeometric functions, Gauss

Introduction

A function of generalised hypergeometry It is possible to express the pFq(a_1,...,a_p;b_1,...,b_q;x) expression as a hypergeometric series, i.e. a series where the ratio between each component is represented as the sum of the ratios of the preceding components. [1]

$$\frac{C_{k+1}}{C_k} = \frac{P(k)}{Q(k)} = \frac{(k+a_1)(k+a_2)\dots(k+a_p)}{(k+b_1)(k+b_2)\dots(k+b_q)(k+1)}x.$$

It is more frequently known as Gauss's hypergeometric function than as the hypergeometric equation since it is one of the most commonly utilised hypergeometric functions, 2F1(a,b;c;x) p=2 q=1 (Gauss 1812, Barnes 1908).Closed forms are occasionally referred to by the term "hypergeometric series," however the word "hypergeometric function" is less frequently used. [2] It has been more important in recent decades to understand basic hypergeometric series, which are used in a wide range of domains such as combinatorial analysis and statistical and quantum physics. They've given the mathematicians a handy tool for combining and subsuming a slew of disparate conclusions into a single, cohesive whole. [3]

For example, taking the (formal) limit $q \to 1$ in the classical hypergeometric context is an extension of the classical hypergeometric functions. [4] Many classical hypergeometric results can be generalised to the basic hypergeometric level. Classical hypergeometric functions are associated to representations of classical groups; whereas, in the case of q-hypergeometric functions, these representations are related to quantum groups. These symbols, known as "infinite" factorials, are defined in terms of the infinite product when |q| is less than 1.

$$(a;q)_\infty = \prod_{r=0}^{\infty}(1-aq^r).$$

The Gaussian hypergeometric function 2F1(a, b; c; z) can be represented by an infinite series, where | z |< 1. [6]

$$2F1(a,b;c;z) = \sum_{k=0}^{\infty}\left[\frac{(a)_k(b)_k}{(c)_k}\right]\frac{z^k}{k!}$$

Z is GHF's GHF variable in the form of an a-b-c function parameter. It doesn't matter how simple or complex the other three parameters are in relation to z. It is an infinite-series representation of the GHFs that is convergent for any a, B, and C (assuming c is neither negative nor zero, as in Seaborn 1991) and (Whittaker and Watson, 1992) for real −1 < z < 1 (or | z | < 1), and for z = ±1 if c > a+b. [7]

GHF can be referred to as

$$2F1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)k!}z^k$$

The following equation defines the gamma function Γ (n) for n > 0.

$$\Gamma(n) = \int_0^{\infty} t^{n-1}e^{-t}\,dt \qquad (for\ n > 0)$$

Research Methodology

The information gathered for this research is secondary in nature, having been drawn from a variety of previously published sources. The information used to prepare this report was culled from a variety of reliable online sources. A research methodology is a universal way to addressing a study subject through data collection, data evaluation, and results based on the findings of the study. A research technique is a plan for carrying out a research study. Research is the methodical collection and analysis of data for the advancement of knowledge in any field.

Result and Discussion

It is possible to see convergence for $pFq(a_1,a_2,..a_p;b_1,b_2,...b_q;z)$ of z when $p \le q$, divergence for |z|>1, and convergence for |z|<1 in case p=q+1, but the hypergeometric series diverges in case p>q+1 and convergence for $z \neq 0$ in case p>q+1 diverges for $z \neq 0$ in case p>q+1 (and the series does not terminate).

Proof :

By factorial properties, we have $\left|\frac{t_{k+1}}{t_k}\right| = \frac{|z|k^{p-q-1}(1+|a_1|/k)...(1+|a_p|/k)}{|(1+|b_1|/k)...(1+|b_q|/k)(1+1/k)|}$

Assuming that |Z| and the relationships between p-q-1=0 ,p-q-1<0 and p-q-1>0 in K^{p-q-1} are valid, the given convergence requirements will be met using the ratio test."

In this case, the hypergeometric function ${}_{q+1}F_q$ is convergent to the unit circle, where the parametric excess is defined as $\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i$

A theorem of Bell's identifies the hypergeometric function and its special functions as having a mutually exclusive relationship (1968) **[8]**
Theorem:

(1) $P_n(x) = 2F1\left(-n, n+1; 1; \frac{1-x}{2}\right)$

$\Leftrightarrow legendre\ polynomial.$

(2) $P^{m}_{n}(x) = \frac{(n+m)!}{(n-m)!} \frac{(1-x^2)^{m/2}}{2^m\ m!} \times 2F1\left(m-n, m+n+1; m+1; \frac{1-x}{2}\right)$

$\Leftrightarrow associated\ legendre\ polynomial.$

(3) $J_n(x) = \frac{e^{-ix}}{n!}\left(\frac{x}{2}\right)^n\ 1F1\left(n+\frac{1}{2}; 2n+1; 2ix\right)$

$\Leftrightarrow Bessel\ function\ of\ the\ 1st\ kind.$

(4) $H_{2n}(x) = (-1)^n \frac{(2n)!}{n!}\ 1F1\left(-n; \frac{1}{2}; x^2\right)$

$$H_{2n+1}(x) = (-1)^n \frac{2(2n+1)!}{n!} z\ 1F1\left(-n; \frac{3}{2}; x^2\right)$$

$\Leftrightarrow Hermitr\ polynomials.$

(5) $L_n(x) = 1F1(-n;;1;z)$

$\Leftrightarrow Laguerre\ polynomial.$

$$L^{k}_{n}(x) = \frac{\Gamma(n+k+1)}{n!\ \Gamma(k+1)} \times 1F1(-n;;1;z)$$

$\Leftrightarrow Associated\ Laguerre\ polynomial.$

(6) $T_n(x) = 2F1(-n;n;\frac{1}{2};\frac{1-x}{2})$

$$U_n(x) = \sqrt{1-x^2}n\ 2F1(-n+1;n+1;\frac{3}{2};\frac{1-x}{2})$$

$\Leftrightarrow Tchebychev/Tchebycheff/Tschebycheff\ polynomial.$

(7) $C^{\lambda}_{n}(x) = \frac{\Gamma(n+2\lambda)}{n!\ \Gamma(2\lambda)} 2F1(-n;n+2\lambda;\lambda+\frac{1}{2};\frac{1-x}{2})$

$\Leftrightarrow Gegenbauer\ polynomial.$

(8) $P^{\alpha,\beta}_{n}(x) = \frac{\Gamma(n+\alpha+1)}{n!\ \Gamma(\alpha)} 2F1(-n;n+\alpha+\beta+1;\alpha+1;\frac{1}{2}(x-1))$

$\Leftrightarrow Jacobi\ polynomial.$

Expanding the hypergeometric function into a series and comparing it to a known series for the given function proves the outcome in each case. For the Legendre polynomial, we may demonstrate this by proving result (1). [9]

Proof of (1):

$$P_n(x) = 2F1\left(-n, n+1; 1; \frac{1-x}{2}\right)$$

$\Leftrightarrow$ *legendre polynomial.*

By definition

$$2F1\left(-n, n+1; 1; \frac{1-x}{2}\right) = \sum_{r=0}^{\infty} \frac{(-n)_r (n+1)_r}{(1)_r (1)_r} \left(\frac{1-x}{2}\right)^r$$

Where

$$(-n)_r = (-n)(-n+1)(-n+2)\ldots(-n+r-1)$$
$$= (-1)^r n(n-1)(n-2)\ldots(n-r+1)$$
$$= (-1)^r \frac{n!}{(n-r)!}$$

Also,

$$(n+1)_r = (n+1)(n+2)\ldots(n+r) = \frac{(n+r)!}{n!}$$

and $(1)_r$ 1 2 r = · ··· =! r r so that we have:

$$2F1\left(-n, n+1; 1; \frac{1-x}{2}\right) = \sum_{r=0}^{n} \frac{n!}{(n-r)!} \frac{(n+r)!}{n!} \frac{1}{(r!)^2} \frac{(1-x)^r}{2^r}$$

$$= \sum_{r=0}^{n} \frac{(-1)^r}{2^r} \frac{(n+r)!}{(n-r)!\,(r!)^2} (1-x)^r$$

$$= \sum_{r=0}^{n} \frac{(n+r)!}{2^r (n-r)!\,(r!)^2} (x-1)^r$$

This is the same as $P_n(x)$, thus we can express $P_n(x)$ as a power series in (x − 1) to demonstrate the equality. Expanding Taylor series:[10]

$$P_n(x) = \sum_{r=0}^{n} P_n^{(r)} \frac{(x-1)^r}{r!}$$

where $P_n^{(r)}$ is the rth derivative of Pn(x) evaluated at x = 1

To calculate $P_n^{(r)}(1)$ use the generating function for the Legendre polynomials given by:[11]

$$\frac{1}{(1-2tx+t^2)^{1/2}} = \sum_{n=0}^{\infty} P_n(x) t^n$$

so that by multiplying r times by x, we get:

$$\sum_{n=0}^{\infty} P_n^{(r)}(x)t^n = \frac{d^r}{dx^r}(1-2tx+t^2)^{-1/2}$$

$$= (-2t)^r\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\dots\left(-\frac{1}{2}-r+1\right)(1-2tx+t^2)^{-\frac{1}{2}-r}$$

$$= 2^r t^r \frac{1}{2}\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)\dots\left(\frac{1}{2}+r-1\right)(1-2tx+x^2)^{-\frac{1}{2}-r}$$

$$= t^r\, 1\cdot 3\cdot 5\cdots(2r-1)(1-2tx+t^2)^{-\frac{1}{2}-r}$$

$$= t^r \frac{(2r)!}{2^r\, r!}\,(1-2tx+t^2)^{-\frac{1}{2}-r}$$

Setting x = 1, we get, by the binomial theorem:[12]

$$\sum_{n=0}^{\infty} P_n^{(r)}(1)t^n = \frac{t^r(2r)!}{2^r\, r!}(1-2t+t^2)^{-\frac{1}{2}-r} = \frac{t^r(2r)!}{2^r\, r!}(1-t)^{-1-2r}$$

$$= \frac{t^r(2r)!}{2^r\, r!}\left\{1+(1+2r)t+\frac{(1+2r)(2+2r)}{2!}t^2+(2r+1)(2r+2)(2r+3)\frac{t^3}{3!}+\cdots\right\}$$

$$= \frac{t^r(2r)!}{2^r\, r!}\sum_{s=0}^{\infty}\frac{(2r+s)}{2r!\,s!} = \frac{1}{2^r\ r!}\sum_{s=0}^{\infty}\frac{(2r+s)!}{s!}t^{r+s}$$

Replacing r + s = n, with 0 ⩽ s ⩽ ∞ ⇒ r ⩽ n ⩽ ∞:[13]

$$\sum_{n=0}^{\infty} P_n^{(r)}(1)t^n = \frac{1}{2^r\ r!}\sum_{n=r}^{\infty}\frac{(r+n)!}{(n-r)!}t^n$$

Hence,

$$P_n(r) = \sum_{r=0}^{n} P_n^{(r)}(1)\frac{(x-1)^r}{r!} = \sum_{r=0}^{\infty}\frac{1}{2^r\ r!}\frac{(n+r)!}{(n-r)!}\frac{(x-1)^r}{r!}$$

$$= \sum_{r=0}^{n}\frac{(n+r)!}{(n-r)!}\frac{(x-1)^r}{(r!)^2} = 2F1\left(-n, n+1; 1; \frac{1-x}{2}\right)$$

Conclusion

Applied mathematics has become so reliant on hypergeometric functions that they may be found in a variety of computer systems, including Maple and Mathematica. A big advantage they have is their ability to provide clear responses to questions and their concision. Hypergeometric functions and special functions have been shown to have a direct correlation with one another, according to Bell's theorem

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