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QUANTUM Q-SERIES AND MOCK THETA FUNCTIONS

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ABSTRACT:
Through quantum q-series identities, the fake theta functions and quantum modular forms are connected. Following Lovejoy, quantum q-series identities are such that they hold at dense sets of roots of unity on the boundary but not as an equality between power series inside the unit disc in the classical sense. A number of general multivariable quantum q-series identities are established and they are used to different scenarios concerning universal fake theta functions. Consequently, we surprisingly demonstrate that mock theta functions at roots of unity that are limiting, finite, and universal, and whose infinite counterparts do not converge, are quantum modular. Furthermore, we demonstrate the crucial roles played by these finite limiting universal mock theta functions in modified Ramanujan radial limits.

KEYWORDS:
Mock theta functions, Quantum modular form, q-series, q-hypergeometric series, Basic hypergeometric series, Quantum q-series

INTRODUCTION:

Among the biggest discoveries made by Ramanujan are the fake theta functions. In the last ten years, these functions have finally been grasped in the context of modular shapes, after eight decades of mystery. It is now known that some weight 1/2 harmonic Maass forms include holomorphic sections that correspond to mock theta functions. More broadly, a mock modular form of weight k is the holomorphic portion of a weight k harmonic Maass form. A mixed mock modular form results from permitting the multiplication of a mock modular form by a modular form. In physics, algebra, and number theory, mock modular forms are frequently of the mixed type.In the quantum theory of black holes and wall-crossing phenomenon, for instance, mixed mock modular forms have recently surfaced as characters in the theory of affine Lie superalgebras, as generating functions for exact formulas for the Euler numbers of certain moduli spaces, for Joyce invariants and for linking numbers in 3-manifolds, in relation to other automorphic objects, and in the combinatorial setting of q-series and partitions. [1]

The bond between Hardy and Ramanujan is the tale of mathematics. With two well-known letters, it started and concluded. First, addressed to Hardy in 1913 by Ramanujan, it describes the author as a poor clerk in a shipping office in Madras who has made certain discoveries that "are termed by the local mathematicians as'startling'." Hardy summoned Ramanujan to England for what would turn out to be one of the most well-known mathematical partnerships in history after spending the night with Littlewood and persuading himself that the letter was the creation of a genius rather than a fake.The second letter was written by Ramanujan to Hardy in 1920, three months before he passed away in India at the age of 32. Hardy had returned to India after spending five years in England. Here he emerges momentarily from his sickness and melancholy to tell Hardy with excitement about a new class of functions he has found and is calling "mock theta functions."

Teaser hinting at his theory of mock theta functions can be found in Ramanujan's deathbed letter. It is now established that these functions are effectively the period integrals of weight 3/2 unary theta functions, whose nonholomorphic components are the holomorphic parts of weight 1/2 harmonic weak Maass forms1. There are numerous uses for this insight. [2]

Even though there are a ton of applications for the theory of weak Maass forms in various branches of mathematics, the deeper context of the ideas in Ramanujan's final letter to Hardy remains unclear. visit the initial motivations and assertions made by Ramanujan. The last letter he wrote outlines the asymptotic features of the modular "Eulerian" series near the roots of unity. In his question, Ramanujan seeks to know if other asymptotically identical Eulerian series are invariably the product of a modular theta function and a function that is O (1) at all roots of unity. He composes:

‘The answer is it is not necessarily so . . . I have not proved rigorously that it is not necessarily so . . . But I have constructed a number of examples . . . ’ In fact, Ramanujan’s sole example and claim pertain to his third-order mock theta function f (q).

As q approaches an even-order 2k root of unity, we have

$$f(q) - (-1)^k(1-q)(1-q^3)(1-q^5)\cdots(1-2q+2q^4-\cdots) = O(1).$$

$$f(q) - (-1)^k(1-q)(1-q^5)\ldots(1-2q+2q^4-\cdots) = 0(1).$$

We have verified Ramanujan's speculation. For Ramanujan's mock theta functions, there are no weakly holomorphic modular forms that precisely eliminate the singularities. Ramanujan provides only one particular example, which concerns the q-hypergeometric functions. [3]

$$f(q) := 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \cdots. \quad (1.1)$$

$$f(q) \coloneqq 1 + \frac{q}{(1-q)^2} + \frac{q^4}{(1-q)^2(1-q^2)^2} + \cdots$$

For |q| < 1 and those odd-order roots of unity q, this function converges.

There are exponential singularities in f (q) for even-order roots of unity. As an illustration, as q → −1, we have

f (−0.994) ~ −1.08 · 10, f (−0.996) ~ −1.02 · 10, f (−0.998) ~ −6.41 · 10, To cancel the exponential singularity at $q = -1$, Ramanujan found the function $b(q)$, which is modular (Here, $q^{-\frac{1}{24}}b(q)$ is modular with respect to z, where $q := e^{2\pi iz}$.) up to multiplication by $q^{-\frac{1}{24}}$, defined in his notation by

$$b(q) := (1-q)(1-q^3)(1-q^5)\cdots(1-2q+2q^4-\cdots). \qquad (1.2)$$

To cancel the exponential singularity at q=-1, Ramanujan found the function b(q), which is modular (Here, $q - \frac{1}{24}b(q)$ is modular with respect to z, where $q := e^{2\pi iz}$.)

$$b(q) := (1-q)(1-q^3)(1-q^5) \ldots (1-2q+2q^4-\cdots).$$

Looking at it this way, lim q→−1(f (q) + b(q)) = 4. More generally, Ramanujan asserted that, as q approaches an even-order 2k root of unity,

$$f(q) - (-1)^k b(q) = O(1).$$

(1.3)

$$f(q) - (-1)^k b(q) = O(1)$$

As q moves radially toward ζ inside the unit disk, if ζ is a primitive even-order 2k root of unity, we get that

$$\lim_{q\to\zeta}\left(f(q) - \left(-1\right)^k b(q)\right) = -4.\sum_{n=0}^{k-1}(1+\zeta)^2(1+\zeta^2)^2 \ldots (1+\zeta^n)^2\zeta^{n+1}$$

$$\lim_{q\to\zeta}(f(q) - (-1)^k b(q)) = -4\cdot\sum_{n=0}^{k-1}(1+\zeta)^2(1+\zeta^2)^2\cdots(1+\zeta^n)^2\zeta^{n+1}.$$

EXAMPLE. Since empty products equal 1, Theorem 1.1 confirms that lim q→-1 (f (q) + b(q)) = 4.

For k = 2, Theorem 1.1 gives limq→i (f (q) - b(q)) = 4i. The table below nicely illustrates this fact:

q	$0.992i$	$0.994i$	$0.996i$
$f(q)$	$\sim 1.9\cdot 10^6 - 4.6\cdot 10^6 i$	$\sim 1.6\cdot 10^8 - 3.9\cdot 10^8 i$	$\sim 1.0\cdot 10^{12} - 2.5\cdot 10^{12} i$
$f(q)-b(q)$	$\sim 0.0577 + 3.855i$	$\sim 0.0443 + 3.889i$	$\sim 0.0303 + 3.924i$

q	0.992i	0.994i	0.996i
f(q)	~1.9.10^6-4.6.10^{6}i	~1.6.10^8-3.9.10^{8}i	~1.0.10^{12}-2.5.10^{12}i
f(q)-b(q)	~0.0577+3.855i	~0.0443+3.889i	~0.0303+3.924i

It is simple to compute the well-defined values of f (q) at odd-order roots of unity straight from (1.1).

Surprisingly, Theorem 1.1 turns out to be a specific case of a much more general theorem that connects two of the most well-known q-series in the theory of partitions. To be more specific, we need the q-hypergeometric series U(w; q), the Andrews–Garvan crank function C(w; q), and Dyson's rank function R(w; q). Among the most significant generating functions in the theory of partitions are the q-series R(w; q) and C(w; q). These well-known series are important to the study of congruences of integer partitions.

To define these series, throughout we let $(a;q)_0 := 1$ and

$$(a;q)_n := \begin{cases} (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}) & \text{if } n \in \mathbb{Z}^+, \\ (1-a)(1-aq)(1-aq^2)\cdots & \text{if } n = \infty. \end{cases}$$

Dyson's rank function is given by

$$R(w;q) = \sum_{n=0}^{\infty} \sum_{m\in\mathbb{Z}} N(m,n) w^m q^n := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(wq;q)_n \cdot (w^{-1}q;q)_n}. \quad (1.4)$$

To define these series, throughout we let $(a;q)_0$:=1 and

$$(a;q)_n := \begin{Bmatrix} (1-a)(1-aq)(1-aq^2)\ldots(1-aq^{n-1}) & if\ n \in Z^+ \\ (1-a)(1-aq)(1-aq^2)\ldots & if\ n = \infty \end{Bmatrix}$$

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The number of partitions of n with rank m in this case is N (m, n), where a partition's rank is equal to the sum of its parts less the greatest part. It is known that R (w; q) is a fake theta function (up to a power of q) if w 6 = 1 is a root of unity. [4]

The function of the Andrews-Garvan crank is defined by

$$C(w;q) = \sum_{n=0}^{\infty} \sum_{m\in\mathbb{Z}} M(m,n) w^m q^n := \frac{(q;q)_\infty}{(wq;q)_\infty \cdot (w^{-1}q;q)_\infty}. \quad (1.5)$$

$$C(w;q) = \sum_{n=0}^{\infty} \sum_{m\in Z} M(m,n) w^m q^n := \frac{(q;q)_\infty}{(wq;q)_\infty\ (w^{-1}q;q)_\infty} \quad (1.5)$$

The number of partitions of n with crank m is represented here by M(m, n) [5]. C(w; q) is a modular form (to a power of q) for roots of unity w.

The definition of this q-hypergeometric series is

$$U(w;q) = \sum_{n=0}^{\infty} \sum_{m\in\mathbb{Z}} u(m,n)(-w)^m q^n := \sum_{n=0}^{\infty} (wq;q)_n \cdot (w^{-1}q;q)_n q^{n+1}. \quad (1.6)$$

$$U(w;q) = \sum_{n=0}^{\infty} \sum_{m\in Z} u(m,n)(-w)^m q^n := \sum_{n=0}^{\infty} (wq;q)_n\ (w^{-1}q;q)_n q^{n+1} \quad (1.6)$$

Here, the number of strongly unimodal sequences of size n with rank m is denoted as u (m, n) [5].

INDEFINITE THETA SERIES:

Let Q(x) =1/2 <x,x) be the corresponding quadratic form and let (,) be a Z-valued bilinear form on Z r. It is a traditional fact that the theta series O(r) = £, € Z ^ Q (1 /) for positive definite Q is a modular form of weight r/2 (and of known level and character), or more

generally 6 a, 6 (r) = £ „ G Z. + a e 2 v i < h^ q ^ for any a, b G Q r. Although Siegel is well-known for his theory of non-holomorphic theta series, there is currently no accepted method for obtaining holomorphic functions with arithmetic Fourier coefficients that exhibit any type of modular transformation behavior when dealing with indefinite theta series.

The theta series Oa,b as defined above is divergent for Q indeterminate because all of its terms occur with infinite multiplicity (since there is an infinite group of units permuting the terms) and are unbounded because there are vectors v G Z r with Q(v) < 0). By limiting the summing to the set of lattice points located between two carefully selected hyperplanes in W, we can, however, make it convergent. Let C be one of the two elements of the double cone {x G R r | Q(x) < 0}, and define for a, b G Q r and c, c' G C. To put it another way.

$$\Theta_{a,b}^{c,c'}(\tau) = \sum_{\nu\in\mathbb{Z}^r+a} \left(\operatorname{sgn}(\langle c,\nu\rangle) - \operatorname{sgn}(\langle c',\nu\rangle)\right) e^{2\pi i\langle b,\nu\rangle} q^{Q(\nu)}.$$

$$\ominus_{a,b}^{c,c'}(\tau) = \sum_{v\in Z^r+a} \left(sgn\big((c,v)\big) - sgn\left((c',v)\right)\right) e^{2\pi i(b,v)} q^{Q(v)}.$$

Of course, this series is not generally modular, but it does define a holomorphic function of r as it now only consists of positive powers of q and is absolutely convergent, even if this isn't immediately apparent. Zwegers presents the updated function as a fix for this.

$$\widehat{\Theta}_{a,b}^{c,c'}(\tau) = \sum_{\nu\in\mathbb{Z}^r+a} \left(E\left(\frac{\langle c,\nu\rangle\sqrt{y}}{\sqrt{-Q(c)}}\right) - E\left(\frac{\langle c',\nu\rangle\sqrt{y}}{\sqrt{-Q(c')}}\right)\right) e^{2\pi i\langle b,\nu\rangle} q^{Q(\nu)} \qquad (y = \Im(\tau))$$

with $E(z)$ as in Theorem 2.1. Then from the relation $E(x) = \operatorname{sgn}(x)\,(1-\beta(x^2))$ we get $\widehat{\Theta}_{a,b}^{c,c'}(\tau) = \Theta_{a,b}^{c,c'}(\tau) - \Phi_{a,b}^{c}(\tau) + \Phi_{a,b}^{c'}(\tau)$ with

$$\Phi_{a,b}^{c}(\tau) = \sum_{\nu\in\mathbb{Z}^r+a} \operatorname{sgn}(\langle c,\nu\rangle)\,\beta\left(\frac{\langle c,\nu\rangle^2 y}{-Q(c)}\right) e^{2\pi i\langle b,\nu\rangle} q^{Q(\nu)}$$

$$\widehat{\ominus}_{a,b}^{c,c'}(\tau) = \sum_{v\in Z^r+a} \left(E\left(\frac{\langle c,v\rangle\sqrt{y}}{\sqrt{-Q(c)}}\right) - E\left(\frac{\langle c',v\rangle\sqrt{y}}{\sqrt{-Q(c')}}\right)\right) e^{2\pi i\langle b,v\rangle} q^{Q(v)} \qquad (y = \Im(\tau))$$

With E(z) as in Theorem 2.1 Then from the relation E(x)= sgn(x)(1-β(x^2)) we get

$$\widehat{\ominus}_{a,b}^{c,c'}(\tau) = \widehat{\ominus}_{a,b}^{c,c'}(\tau) - \Phi_{a,b}^{c}(\tau) + \Phi_{a,b}^{c'}(\tau) with$$

$$\Phi_{a,b}^{c}(\tau) = \sum_{v\in Z^r+a} sgn(\langle c,v\rangle)\beta\left(\frac{\langle c,v\rangle^2 y}{-Q(c)}\right) e^{2\pi i\langle b,v\rangle} q^{Q(v)}$$

Given c as a member of C f l Q r, $1^(T) is a finite linear combination. J2j Rj(T) Qj(T) where each Rj (r) is a sum of the same kind that happened in §2 as the necessary adjustment to convert mock theta functions into modular functions

$$R_j(\tau) = \textstyle\sum_{n\in\mathbb{Z}+\alpha_j} \operatorname{sgn}(n)\,\beta(4\kappa_j n^2 y)\,q^{-\kappa_j n^2}$$

$$R_j(\tau) = \sum n \in Z + a_j \; sgn(n)\beta(4kjn^2 y)\, q^{-kj\,n^2}$$

Weight (r — l) / 2 has a holomorphic modular form for some ctj G Q and Kj G Q > O), where each 0j (r) is an ordinary theta series linked to the quadratic form QKc)1-. [6]

Bailey pairs:

Thus, we can now comprehend the modular behavior of a q-series if we can explain it in terms of Appell-Lerch series or indefinite theta functions. Bailey pair theory is our main technique for demonstrating identities of q-series. There were still only about 45 q-series that were recognized as mock theta functions ten years after Zwegers' breakthrough. Let us investigate the consequences of attempting to generate families of counterfeit theta functions using the Bailey chain.

Consider the Bailey pair with respect to 1.

$$\alpha_n = \begin{cases} 1, & \text{if } n = 0, \\ \frac{4(-1)^n q^{\binom{n+1}{2}}}{(1+q^n)}, & \text{otherwise,} \end{cases}$$

and

$$\beta_n = \frac{1}{(-q)_n^2}.$$

$$\alpha_n = \begin{Bmatrix} 1, \\ \frac{4(-1)^2 q \binom{n+1}{2}}{(1+q^n)} \end{Bmatrix} \begin{matrix} if\ n = 0, \\ otherwise, \end{matrix}$$

And

$$\beta_n = \left\{\frac{1}{(-q)_n^2}\right\}$$

This can be inferred from Slater's work.

A Bailey chain iteration where b, c → ∞ at each step yields

$$\sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{n_k^2+n_{k-1}^2+\cdots+n_1^2}}{(q)_{n_k-n_{k-1}} \cdots (q)_{n_2-n_1}(-q)_{n_1}^2} = \frac{2}{(q)_\infty} \sum_{n \in \mathbb{Z}} \frac{q^{kn^2+\binom{n+1}{2}}(-1)^n}{(1+q^n)}.$$

$$\sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{n_k^2+n_{k-1}^2+\cdots+n_1^2}}{(q)n_k - n_{k-1} \ldots (q)_{n2-n1}(-q)_{n1}^2} = \frac{2}{(q)_\infty} \sum_{n \in Z} \frac{q^{kn^2} + \binom{n+1}{2}(-1)^n}{(1+q^n)}$$

The case k = 1 is the Appell-Lerch type series for f (q).

The case k ≥ 2 is not a mock theta function.

For example, when k = 2 we have the identity

$$\sum_{n_2 \geq n_1 \geq 0} \frac{q^{n_2^2+n_1^2}}{(q)_{n_2-n_1}(-q)_{n_1}^2} = -\frac{2}{(q^2,q^3;q^5)_\infty}\chi(q) + \frac{2}{(q,q^4;q^5)_\infty}X(q) - \frac{(q)_\infty}{(-q)_\infty^2},$$

$$\sum_{n_2 \geq n_1 \geq 0} \frac{q^{n_2^2+n_1^2}}{(q)_{n2-n1}(-q)_{n1}^2} = -\frac{2}{(q^2,q^3;q^5)_\infty}x(q) + \frac{2}{(q,q^4;q^5)_\infty}x(q) - \frac{(q)_\infty}{(-q)_\infty^2},$$

where χ and X are two tenth order mock theta functions,

$$\chi(q) = \sum_{n\geq 0} \frac{(-1)^n q^{(n+1)^2}}{(-q)_{2n+1}},$$

$$X(q) = \sum_{n\geq 0} \frac{(-1)^n q^{n^2}}{(-q)_{2n}}.$$

$$x(q) = \sum_{n\geq 0} \frac{(-1)^n q^{(n+1)^2}}{(-q)_{2n+1}}$$

$$x(q) = \sum_{n\geq 0} \frac{(-1)^n q^{n^2}}{(-q)_{2n}}$$

This is a mixed mock modular form.

REVIEW OF LITERATURE:

In particular, Watson verified all of Ramanujan's identities and asymptotic expansions for the functions of order 3, taking into account the mock theta functions from Ramanujan's previous letter. In order to accomplish this, he first developed a number of new identities, some of which were not all that novel, as it turned out when Ramanujan's "lost notebooks" were found, which connected the mock theta functions to q hypergeometric series in a much simpler form. The identity is such an example. [7]

It turns out, according to Zwegers, Bruinier-Funke, and others like Bringmann-Ono and Zagier, that the mock theta functions are instances of mock modular forms, which are holomorphic components of harmonic Maass forms and contemporary analogues of both regular Maass forms and modular forms. We have also been able to better understand the concept of the order of a fake theta function thanks to this context. For additional context and details regarding these elements of the simulated theta functions. [8][9]

OBJECTIVES:

- To understand therotical study of Ramanujan’s greatest discoveries mock theta function.
- To illustrate and resulting with Dyson’s rank function, Zwegers and bailey pairs

RESEARCH METHODOLOGY:

The overall design of this study was exploratory. The research paper is an effort that is based on secondary data that was gathered from credible publications, the internet, articles, textbooks, and newspapers. The study's research design is primarily descriptive in nature.

RESULT AND DISCUSSION:

Consequently, we demonstrate that Ramanujan's assertion is a particular instance of a more general outcome. There is nothing strange about the constants that Ramanujan implies. They appear in Zagier's "quantum modular forms" theory. We give explicit closed formulations for these "radial limits" in terms of values of a "quantum" q-hypergeometric function, which forms the basis of a new connection between the Andrews-Garvan crank modular form and Dyson's rank mimic theta function. In this vein, we demonstrate that the functions known as the Rogers–Fine false ϑ-functions, which are poorly understood in the context of modular form theory, specialize to quantum modular forms.

Suppose that k, K ≥ 1, 0 ≤ ℓ < k and 0 ≤ m < K.

(1) The sequences (α(k,K ,ℓ)n , β(k,K ,ℓ)n) form a Bailey pair relative toq, where

$$\alpha_n^{(k,K,\ell)} = \frac{q^{(K+1)n^2+Kn}(1-q^{2n+1})}{(1-q)} \sum_{j=-n}^{n} (-1)^j q^{-((2k+1)j^2+(2\ell+1)j)/2}$$

and

$$\beta_n^{(k,K,\ell)} = \sum_{n \geq n_{k+K-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{\sum_{i=1}^{K-1} n_{k+i}(n_{k+i}+1)+\binom{n_k+1}{2}-\sum_{i=1}^{k-1} n_i n_{i+1}-\sum_{i=1}^{\ell} n_i}(-1)^{n_k}}{(q)_{n-n_{k+K-1}}(q)_{n_{k+K-1}-n_{k+K-2}}\cdots(q)_{n_2-n_1}(q)_{n_1}}.$$

$$\alpha_n^{(k,K,l)} = \frac{q^{(K+1)n^2+Kn}(1-q^{2n+1})}{(1-q)} \sum_{j=-n}^{n} (-1)^j q - ((2k+1)j^2 + (2l+1)j/2$$

And

$$\sum_{n \geq n_{k+K-1 \geq \cdots \geq n_1 \geq 0}} \frac{q \sum_{i=1}^{k-1} n_{k+i}(ni+1) + \frac{nK+1}{2} - \sum_{i=1}^{k-1} n_i n_{i+1 \sum_{i=1}^{l} ni} (-1)^{nk}}{(q)_{n-nk+K-1}(q)_{nk+K-1^{-n}{}_{k+K-2}\ldots(q)_{n2-n1}(q)_{n1}}}$$

2) The sequences (α(k,K ,ℓ,m)n , β(k,K ,ℓ,m)n) form a Bailey pairrelative to 1, where

$$\alpha_n^{(k,K,\ell,m)} = q^{(K+1)n^2+(m+1)n} \sum_{j=-n}^{n} (-1)^j q^{-((2k+1)j^2+(2\ell+1)j)/2}$$
$$- \chi(n \neq 0) q^{(K+1)n^2-(m+1)n} \sum_{j=-n+1}^{n-1} (-1)^j q^{-((2k+1)j^2+(2\ell+1)j)/2}$$

and

$$\beta_n^{(k,K,\ell,m)} = \sum_{n \geq n_{k+K-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{\sum_{i=1}^{K-1} n_{k+i}^2 + \sum_{i=1}^{m} n_{k+i} + \binom{n_k+1}{2} - \sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell} n_i} (-1)^{n_k}}{(q)_{n-n_{k+K-1}} (q)_{n_{k+K-1}-n_{k+K-2}} \cdots (q)_{n_2-n_1} (q)_{n_1}}.$$

$$\alpha_n^{(k,K,l,m)} = q^{(K+1)_n 2+(m+1)n} \sum_{j=-n}^{n} (-1)^j q^{-((2k+1)j^2+(2l+1)j)/2}$$
$$- x(n \neq 0) q^{(k+1)n^2-(m+1)^n} \sum_{j=-n+1}^{n-1} (-1)^j q^{-((2k+1)j^2+(2l+1)j)/:}$$

And

$$\beta_n^{(k,K,l,m)} = \sum_{n \geq nk+K-1 \geq \cdots \geq n1 \geq 0} \frac{q \sum_{i=1}^{k-1} n_{i=1}^2 \sum_{i=1}^{m} (n_{k+i} + \binom{nK+1}{2}) - \sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{l} n_i (-1)^{nk}}{(q)_{n-nk+K-1} (q)_{nk+K-1-^n k+K-2} \cdots (q)_{n2-n1} (q)_{n1}}$$

For each of the four indefinite quadratic forms previously discussed, there is an equivalent result. Bailey lemma, Bailey lattice, Bailey lattice replacement, and dual Bailey pairings are all used in the proof. You can get multisum fake theta functions from these pairs. Assume, for illustration, that $k > 1$, $0 \leq \ell < k$, and $0 \leq m \leq k$.[10]

CONCLUSION:

Among the most fascinating items found in Ramanujan's contribution to mathematics are the theorems presented in this paper. In fact, the mock theta functions, Dyson's rank, the Andrews-Garvan crank, and early instances of quantum modular forms.

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